

PRIMITIVITY OF UNITAL FULL FREE PRODUCTS OF RESIDUALLY FINITE DIMENSIONAL C^* -ALGEBRAS

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ABSTRACT. A C^* -algebra is called primitive if it admits a faithful and irreducible $*$ -representation. We show that if A_1 and A_2 are separable, unital, residually finite dimensional C^* -algebras satisfying $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$, then the unital C^* -algebra full free product, $A = A_1 * A_2$, is primitive. It follows that A is antiliminal, it has an uncountable family of pairwise inequivalent irreducible faithful $*$ -representations and the set of pure states is w^* -dense in the state space.

1. INTRODUCTION

A C^* -algebra is called primitive if it admits a faithful and irreducible $*$ -representation. Thus the simplest examples are matrix algebras. A nontrivial example, shown independently by Choi and Yoshizawa, is the full group C^* -algebra of the free group on n elements, $2 \leq n \leq \infty$, see [4] and [15]. In [10], Murphy gave numerous conditions for primitivity of full group C^* -algebras. More recently, T. Å. Omland showed in [11] that for G_1 and G_2 countable amenable discrete groups and σ a multiplier on the free product $G_1 * G_2$, the full twisted group C^* -algebra $C^*(G_1 * G_2, \sigma)$ is primitive whenever $(|G_1| - 1)(|G_2| - 1) \geq 2$.

We prove that given two nontrivial, separable, unital, residually finite dimensional C^* -algebras A_1 and A_2 , their unital C^* -algebra full free product $A_1 * A_2$ is primitive except when $A_1 = \mathbb{C}^2 = A_2$. The methods used are essentially different from those in [10], [2], [1] and [11] but do rely on Exel and Loring's result [7] that $A_1 * A_2$ is itself residually finite dimensional. Roughly speaking, we first show that if $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$, then there is an abundance of irreducible finite dimensional $*$ -representations and later, by means of a sequence of approximations, we construct an irreducible and faithful $*$ -representation.

Date: June 19, 2012.

2000 Mathematics Subject Classification. 46L09 (46L05).

Key words and phrases. Primitive C^* -algebra, Full free product.

Research supported in part by NSF grant DMS-0901220.

The paper is divided as follows. Section 2 recalls some facts about $*$ -automorphisms of finite dimensional C^* -algebras. Section 3 recalls some known result on Lie groups that will be used later. Section 4 is fully devoted in proving Theorem 4.1 which is about perturbing a pair of proper unital C^* -subalgebras of a matrix algebra in such a way that they have trivial intersection. Theorem 4.1 is the cornerstone for the rest of the results in the paper. Lastly, section 5 contains the proof of the main theorem about primitivity and some consequences.

Notation 1.1. Given a Hilbert space H , we denote the set of bounded linear operators by $\mathbb{B}(H)$ and the set of compact operators by $\mathbb{K}(H)$.

For a unital C^* -algebra A , $*$ -SubAlg(A) denotes the set of all unital C^* -subalgebras of A and $\mathbb{U}(A)$ denotes the set of unitary elements of A . For simplicity, given a Hilbert space H we write $\mathbb{U}(H)$ instead of $\mathbb{U}(\mathbb{B}(H))$.

By $\text{Aut}(A)$ we denote the set of $*$ -automorphisms of A . For u in $\mathbb{U}(A)$ we let $\text{Ad } u$ denote the $*$ -automorphism of A given by $\text{Ad } u(x) = u x u^*$. The set of all $*$ -automorphisms of the form $\text{Ad } u$, for some u , is called the set of inner automorphism and it is denoted by $\text{Inn}(A)$.

For a unital C^* -algebra A , $C(A)$ denotes its center. In other words

$$C(A) = \{x \in A : xa = ax \text{ for all } a \in A\}.$$

For a positive integer n , M_n denotes the set of $n \times n$ matrices over \mathbb{C} and S_n denotes the permutation group of the set $\{1, \dots, n\}$.

2. $*$ -AUTOMORPHISMS OF FINITE DIMENSIONAL C^* -ALGEBRAS

By a $*$ -automorphism of a C^* -algebra we mean a bijective map, from the algebra onto itself, that is linear and preserves products and adjoints.

In this section we recall some basic results concerning $*$ -automorphisms of finite dimensional C^* -algebras and in particular a precise algebraic relation between the group of $*$ -automorphism and the subgroup of inner $*$ -automorphisms.

Any $*$ -homomorphism from a simple C^* -algebra is either zero or injective (since its kernel is an ideal). Even more, any non-zero $*$ -endomorphism of a finite dimensional simple C^* -algebra is a $*$ -automorphism. Indeed, any such $*$ -endomorphism is injective and thus it is bijective (by finite dimensionality) and a straightforward computation shows its inverse is a $*$ -endomorphism. As a consequence any $*$ -automorphism of a finite dimensional C^* -algebra moves, without breaking, each one of its simple C^* -subalgebras (we may think

this as blocks) with the same dimension. Thus modulo an inner $*$ -automorphism, a $*$ -automorphism is just a permutation. We make the last statement precise with the following two propositions.

Proposition 2.1. *Let B be a finite dimensional C^* -algebra and assume B decomposes as*

$$\bigoplus_{j=1}^J B_j$$

and there is a positive integer n such that all B_j are $$ -isomorphic to M_n .*

Fix $\{\beta_j : B_j \rightarrow M_n\}_{1 \leq j \leq J}$ a set of $$ -isomorphisms.*

(1) *For a permutation σ in S_J define $\psi_\sigma : B \rightarrow B$ by*

$$\psi_\sigma(b_1, \dots, b_J) = (\beta_1^{-1} \circ \beta_{\sigma^{-1}(1)}(b_{\sigma^{-1}(1)}), \dots, \beta_J^{-1} \circ \beta_{\sigma^{-1}(J)}(b_{\sigma^{-1}(J)}))$$

Then ψ_σ lies in $\text{Aut}(B)$ and the map $\sigma \mapsto \psi_\sigma$ defines a group embedding of S_J into $\text{Aut}(B)$.

(2) *Every element α in $\text{Aut}(B)$ factors as*

$$\left(\bigoplus_{j=1}^J \text{Ad } u_j\right) \circ \psi_\sigma$$

for some permutation σ in S_J and unitaries u_j in $\mathbb{U}(B_j)$.

(3) *There is a exact sequence*

$$0 \rightarrow \text{Inn}(B) \rightarrow \text{Aut}(B) \rightarrow S_J \rightarrow 0.$$

So far we have consider C^* -algebras with only one type of block subalgebra, so to speak. Next proposition shows that a $*$ -automorphism can not mix blocks of different dimensions. As a consequence, and along with Proposition 2.1, we get a general decomposition of $*$ -automorphisms of finite dimensional C^* -algebras.

Proposition 2.2. *Let B be a finite dimensional C^* -algebra. Decompose B as*

$$\bigoplus_{i=1}^I \bigoplus_{j=1}^{J_i} B(i, j)$$

where for each i , there is a positive integer n_i such that $B(i, j)$ is isomorphic to M_{n_i} for all $1 \leq j \leq J_i$, i.e. we group subalgebras that are isomorphic to the same matrix algebra, and where $n_1 < n_2 < \dots < n_I$.

Then any α in $\text{Aut}(B)$ factors as $\alpha = \bigoplus_{i=1}^I \alpha_i$ where

$$\alpha_i : \bigoplus_{j=1}^{J_i} B(i, j) \rightarrow \bigoplus_{j=1}^{J_i} B(i, j)$$

is a $$ -isomorphism.*

3. USEFUL RESULTS FROM LIE GROUPS

In this section we summarize some result that, later on, will be repeatedly used. Definitions and proofs of results mentioned in this section can be found in [9] and [8].

The next two theorems are quite important and will be used in the next section.

Theorem 3.1. *Any closed subgroup of a Lie group is a Lie subgroup.*

Theorem 3.2. *Let G be a Lie group of dimension n and $H \subseteq G$ be a Lie subgroup of dimension k .*

- (1) *Then the left coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical quotient map $\pi : G \rightarrow G/H$, is a fiber bundle, with fiber diffeomorphic to H .*
- (2) *If H is a normal Lie subgroup then G/H has a canonical structure of a Lie group.*

The next proposition is from Corollary 2.21 in [9].

Proposition 3.3. *Let G denote a Lie group and assume it acts smoothly on a manifold M . For $m \in M$ let $\mathcal{O}(m)$ denote its orbit and $\text{Stab}(m)$ denote its stabilizer i.e.*

$$\begin{aligned}\mathcal{O}(m) &= \{g.m : g \in G\}, \\ \text{Stab}(m) &= \{g \in G : g.m = m\}.\end{aligned}$$

The orbit $\mathcal{O}(m)$ is an immersed submanifold of M . If $\mathcal{O}(m)$ is compact, then the map $g \mapsto g.m$, is a diffeomorphism from $G/\text{Stab}(m)$ onto $\mathcal{O}(m)$. (In this case we say $\mathcal{O}(m)$ is an embedded submanifold of M .)

Corollary 3.4. *Let G be a compact Lie group and let K and L be closed subgroups of G . The subspace $KL = \{kl : k \in K, l \in L\}$ is an embedded submanifold of G of dimension*

$$\dim K + \dim L - \dim(L \cap K).$$

Proof. First of all KL is compact. This follows from the fact that multiplication is continuous and both K and L are compact. Consider the action of $K \times L$ on G given by $(k, l).g = kgl^{-1}$. Notice that the orbit of e is precisely KL . By Proposition 3.3, KL is an immersed submanifold diffeomorphic to $K \times L/\text{Stab}(e)$. Since it is compact, it is an embedded submanifold. But $\text{Stab}(e) = \{(x, x) : x \in K \cap L\}$ and we conclude

$$\dim KL = \dim(K \times L) - \dim \text{Stab}(e) = \dim K + \dim L - \dim(K \cap L).$$

□

Proposition 3.5. *Let G be a compact Lie group and let H be a closed subgroup. Let π denote the quotient map onto G/H .*

There are:

- (1) \mathcal{N}_G , a compact neighborhood of e in G ,
- (2) \mathcal{N}_H , a compact neighborhood of e in H ,
- (3) $\mathcal{N}_{G/H}$, a compact neighborhood of $\pi(e)$ in G/H ,
- (4) a continuous function $s : \mathcal{N}_{G/H}(\pi(e)) \rightarrow G$ satisfying
 - (a) $s(\pi(e)) = e$ and $\pi(s(y)) = y$ for all y in $\mathcal{N}_{G/H}(\pi(e))$,
 - (b) The map

$$\begin{aligned} \mathcal{N}_H \times \mathcal{N}_{G/H} &\rightarrow \mathcal{N}_G, \\ (h, y) &\mapsto hs_g(y) \end{aligned}$$

is a homeomorphism.

Proof. Let \mathfrak{g} and \mathfrak{h} denote, respectively, the Lie algebras of G and H . Take \mathfrak{m} a vector subspace such that \mathfrak{g} is the direct sum of \mathfrak{h} and \mathfrak{m} . By Lemmas 2.4 and 4.1 in [8], chapter 2, there are compact neighborhoods $U_{\mathfrak{g}}$, $U_{\mathfrak{h}}$ and $U_{\mathfrak{m}}$ of 0 in \mathfrak{g} , \mathfrak{h} and \mathfrak{m} , respectively, such that the map

$$\begin{aligned} U_{\mathfrak{m}} \times U_{\mathfrak{h}} &\rightarrow U_{\mathfrak{g}}, \\ (a, b) &\mapsto \exp(a)\exp(b) \end{aligned}$$

is an homeomorphism and π maps homeomorphically $\exp(U_{\mathfrak{m}})$ onto a compact neighborhood of $\pi(e)$. Call the latter neighborhood $\mathcal{N}_{G/H}$. Take $\mathcal{N}_G = \exp(U_{\mathfrak{g}})$, $\mathcal{N}_H = \exp(U_{\mathfrak{h}})$ and s the inverse of π restricted to $\exp(U_{\mathfrak{m}})$. □

4. INTERSECTION OF FINITE DIMENSIONAL C^* -ALGEBRAS AND PERTURBATIONS

In this section we fix a positive integer N and, unless stated otherwise, $B_1 \subsetneq M_N$ and $B_2 \subsetneq M_N$ denote proper unital C^* -subalgebras of M_N .

The main purpose of this section is give a proof of the following theorem (recall that for a C^* -algebra A , $C(A)$ denotes its center).

Theorem 4.1. *Assume one of the following conditions holds:*

- (1) $\dim C(B_1) = 1 = \dim C(B_2)$,
- (2) $\dim C(B_1) \geq 2$, $\dim C(B_2) = 1$ and B_1 is $*$ -isomorphic to

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)},$$

- (3) $\dim C(B_1) = 2 = \dim C(B_2)$, B_1 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/2},$$

and B_2 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/(2k)}$$

where $k \geq 2$,

(4) $\dim C(B_1) \geq 2$, $\dim C(B_2) \geq 3$ and, for $i = 1, 2$, B_i is $*$ -isomorphic to

$$M_{N/\dim C(B_i)} \oplus \cdots \oplus M_{N/\dim C(B_i)}.$$

Then

$$\Delta(B_1, B_2) := \{u \in \mathbb{U}(M_N) : B_1 \cap uB_2u^* = \mathbb{C}\}$$

is dense in $\mathbb{U}(M_N)$.

The C^* -algebra uB_2u^* is what we call a perturbation of B_2 by u . With this nomenclature we are trying to prove that, in the cases mentioned above, almost always we can perturb one C^* -subalgebra a little bit in such a way that the intersection with the other one is the smallest possible.

Roughly speaking, the idea behind is to show that the complement of $\Delta(B_1, B_2)$ can be locally parametrized with strictly fewer variables than $\dim \mathbb{U}(M_N) = N^2$. Thus, the complement of $\Delta(B_1, B_2)$ is, topologically speaking, small.

We start with some definitions. The group $\mathbb{U}(B_1)$ acts on $*$ -SubAlg(B_1) via $(u, B) \mapsto uBu^*$ and the equivalence relation on $*$ -SubAlg(B_1) induced by this action will be denoted by \sim_{B_1} . Specifically, we have

$$B \sim_{B_1} C \Leftrightarrow \exists u \in \mathbb{U}(B_1) : uBu^* = C.$$

We denote by $[B]_{B_1}$ the \sim_{B_1} -equivalence class of a subalgebra B in $*$ -SubAlg(B_1).

Notation 4.2. For B in $*$ -SubAlg(B_1) let

$$\begin{aligned} X(B_1, B_2; B) &= \{u \in \mathbb{U}(M_N) : uB_2u^* \cap B_1 = B\}, \\ Y(B_2; B) &= \{u \in \mathbb{U}(M_N) : u^*Bu \subseteq B_2\}, \\ Z(B_1, B_2; [B]_{B_1}) &= \{u \in \mathbb{U}(M_N) : uB_2u^* \cap B_1 \sim_{B_1} B\}. \end{aligned}$$

It is straightforward that the complement of $\Delta(B_1, B_2)$ is precisely the union of the sets $Z(B_1, B_2; [B]_{B_1})$, where B runs over all unital C^* -subalgebras of B_1 and $B \neq \mathbb{C}$. Just for a moment, without being formal, we may think $Z(B_1, B_2; [B]_{B_1})$ as being parametrized by two coordinates. The first one is an algebra \sim_{B_1} -equivalent to B . Hence the first coordinate lives in $[B]_{B_1}$. The second, is a unitary u that realizes the first coordinate as $uB_2u^* \cap B_1$. $X(B_1, B_2; B)$ comes into play in order to parametrize this second coordinate. The problem is that $X(B_1, B_2; B_{B_1})$ is complicated to handle (for instance it may not

be closed). This is way we introduce the friendlier set $Y(B_2; B)$. Good properties about $Y(B_2; B)$ is that it is a closed subset of $\mathbb{U}(M_N)$, in fact we will show it is a finite union of embedded compact submanifolds of $\mathbb{U}(M_N)$, and it contains $X(B_1, B_2; B)$.

The rest of this section is the formalization of the previous idea. In concrete our first goal is to show $[B]_{B_1}$ has a structure of manifold and we are particularly interested in finding its dimension.

Let $\text{Stab}(B_1, B)$ denote the \sim_{B_1} -stabilizer of B i.e.

$$\text{Stab}(B_1, B) = \{u \in \mathbb{U}(B_1) : uBu^* = B\}.$$

Remark 4.3. Given B in $*\text{-SubAlg}(B_1)$ we can endow $[B]_{B_1}$ with a structure of manifold. Indeed, let $\mathbb{U}(B_1)/\text{Stab}(B_1, B)$ denote the set of left-cosets and consider the map

$$\begin{aligned} \beta_B : [B]_{B_1} &\rightarrow \mathbb{U}(B_1)/\text{Stab}(B_1, B), \\ \beta_B(uBu^*) &= u\text{Stab}(B_1, B). \end{aligned}$$

One can check β_B is well defined and bijective. Since $\mathbb{U}(B_1)/\text{Stab}(B_1, B)$ is a manifold, β_B induces a structure of manifold on $[B]_{B_1}$. To avoid ambiguity we have to check the topology does not depend on the representative B . In fact, we will show the topology induced by β_B is the same as the topology induced by the Hausdorff distance.

For C_1 and C_2 in $[B]_{B_1}$ define

$$d_H(C_1, C_2) = \max \left\{ \sup_{x_2} \inf_{x_1} \{\|x_1 - x_2\|\}, \sup_{x_1} \inf_{x_2} \{\|x_1 - x_2\|\} \right\},$$

where x_i is taken in the unit ball of C_i , $i = 1, 2$. Since unit balls of unital C^* -subalgebras of B_1 are compact subsets (in the norm topology), d_H defines a metric on $[B]_{B_1}$. Let τ and τ_H denote, respectively, the topologies on $[B]_{B_1}$ induced by β_B and d_H . We are going to show $\tau = \tau_H$. Consider the identity map $\text{id} : ([B]_{B_1}, \tau) \rightarrow ([B]_{B_1}, \tau_H)$. First we show id is continuous. Since $\mathbb{U}(B_1)/\text{Stab}(B_1, B)$ is endowed with the pull back topology from the quotient map $\pi : \mathbb{U}(B_1) \rightarrow \mathbb{U}(B_1)/\text{Stab}(B_1, B)$ where $\mathbb{U}(B_1)$ is taken with the norm topology, id is continuous if and only if the map

$$\beta_B^{-1} \circ \pi : \mathbb{U}(B_1) \rightarrow ([B]_{B_1}, \tau_H)$$

is continuous. Take $(u_n)_{n \geq 1}$ a sequence in $\mathbb{U}(B_1)$ and a unitary u in $\mathbb{U}(B_1)$ such that $\lim_n \|u_n - u\| = 0$. We need to show

$$\lim_n d_H(\beta_B^{-1} \circ \pi(u_n), \beta_B^{-1} \circ \pi(u)) = \lim_n d_H(u_n Bu_n^*, uBu^*) = 0.$$

Take n_0 such that $\|u_n - u\| < \varepsilon/2$ for all $n \geq n_0$. For any b in the unit ball of B and any $n \geq n_0$, we have

$$\|u_n bu_n^* - ubu^*\| < \varepsilon.$$

Thus, for $n \geq n_0$

$$\sup_{x_2} \inf_{x_1} \|x_1 - x_2\| < \varepsilon$$

and

$$\sup_{x_1} \inf_{x_2} \|x_1 - x_2\| < \varepsilon,$$

where x_2 is taken in the unit ball of $u_n B u_n^*$ and x_1 is taken in the unit ball of $u B u^*$. Hence $\text{id} : ([B]_{B_1}, \tau) \rightarrow ([B]_{B_1}, \tau_H)$ is continuous. Lastly, since id is bijective, $([B]_{B_1}, \tau)$ is compact and $([B]_{B_1}, \tau_H)$ is Hausdorff, we conclude that id is a homeomorphism. Thus $\tau = \tau_H$.

Now that we know $[B]_{B_1}$ is a manifold, we want to find its dimension. Since by construction $[B]_{B_1}$ is diffeomorphic to $\mathbb{U}(B_1)/\text{Stab}(B_1, B)$, $\dim[B]_{B_1} = \dim \mathbb{U}(B_1) - \dim \text{Stab}(B_1, B)$. Thus we only need to find $\dim \text{Stab}(B_1, B)$.

Notation 4.4. Whenever we take commutators they will be with respect to the ambient algebra M_N , in other words for a subalgebra A in $*\text{-SubAlg}(M_N)$

$$A' = \{x \in M_N : xa = ax, \text{ for all } a \text{ in } A\}.$$

Recall that $C(A)$ denotes the center of A i.e.

$$C(A) = A \cap A' = \{a \in A : xa = ax \text{ for all } x \text{ in } A\}.$$

Proposition 4.5. *For any B_1 in $*\text{-SubAlg}(M_N)$ and for any B in $*\text{-SubAlg}(B_1)$, we have*

$$\dim \text{Stab}(B_1, B) = \dim \mathbb{U}(B) + \dim \mathbb{U}(B_1 \cap B') - \dim \mathbb{U}(C(B)).$$

Proof. We'll find a normal subgroup of $\text{Stab}(B_1, B)$, for which we can compute its dimension and that partitions $\text{Stab}(B_1, B)$ into a finite number of cosets. Let G denote the subgroup of $\text{Stab}(B_1, B)$ generated by $\mathbb{U}(B_1 \cap B')$ and $\mathbb{U}(B)$. Since the elements of $\mathbb{U}(B)$ commute with the elements of $\mathbb{U}(B_1 \cap B')$, a typical element of G looks like vw , where v lies in $\mathbb{U}(B)$ and w lies in $\mathbb{U}(B_1 \cap B')$. Taking into account compactness of $\mathbb{U}(B)$ and $\mathbb{U}(B_1 \cap B')$, we deduced G is compact.

Now we show G is normal in $\text{Stab}(B_1, B)$. Take u an element in $\text{Stab}(B_1, B)$. For a unitary v in $\mathbb{U}(B)$ it is immediate that uvu^* lies in $\mathbb{U}(B)$. For a unitary w in $\mathbb{U}(B_1 \cap B')$, the following computation shows uwu^* belongs to $\mathbb{U}(B_1 \cap B')$. For any element b in B we have:

$$(uwu^*)b = uw(u^*bu)u^* = u(u^*bu)wu^* = b(uwu^*),$$

where in the second equality we used u^*bu lies in B . In conclusion uGu^* is contained in G for all u in $\text{St}(B_1, B)$ i.e. G is normal in $\text{Stab}(B_1, B)$.

As a result $\text{Stab}(B_1, B)/G$ is a Lie group. The next step is to show $\text{Stab}(B_1, B)/G$ is finite. Decompose B as

$$B = \oplus_{i=1}^I \oplus_{j=1}^{J_i} B(i, j),$$

where for all i there is k_i such that for $1 \leq j \leq J_i$, $B(i, j)$ is $*$ -isomorphic to M_{k_i} . For the rest of our proof we fix a family, $\beta(i, j) : B(i, j) \rightarrow M_{k_i}$, of $*$ -isomorphisms.

An element u in $\text{Stab}(B_1, B)$ defines a $*$ -automorphism of B by conjugation. As a consequence, Propositions 2.1 and 2.2 imply there are permutations σ_i in S_{J_i} and unitaries v_i in $\mathbb{U}(\oplus_{j=1}^{J_i} B(i, j))$ such that

$$\forall b \in B : ubu^* = v\psi(b)v^* \quad (1)$$

where $v = \oplus_{i=1}^I v_i$ is a unitary in $\mathbb{U}(B)$ and $\psi = \oplus_{i=1}^I \psi_{\sigma_i}$ is a $*$ -automorphism in $\text{Aut}(B)$ (the maps ψ depends on the family of $*$ -isomorphisms $\beta(i, j)$ we fixed earlier). Equation (1) is telling us important information. Firstly, that ψ extends to an $*$ -isomorphism of B_1 and most importantly, this extension is an inner $*$ -automorphism. Fix a unitary U_ψ in $\mathbb{U}(B_1)$ such that $\psi(b) = \text{Ad}U_\psi(b)$ for all b in B (note that U_ψ may not be unique but we just pick one and fix it for rest of the proof). From equation (1) we deduce there is a unitary w in $\mathbb{U}(B_1 \cap B')$ satisfying $u = vU_\psi w$. Since the number of functions ψ , that may arise from (1), is at most $J_1! \cdots J_I!$, we conclude

$$|\text{Stab}(B_1, B)/G| \leq J_1! \cdots J_I!$$

Now that we know $\text{Stab}(B_1, B)/G$ is finite we have $\dim \text{Stab}(B_1, B) = \dim G$, and Corollary 3.4 gives the result. \square

From Proposition 4.5 and Remark 4.3, we get the following corollary.

Corollary 4.6. *For any B_1 in $*$ -SubAlg(M_N) and any B in $*$ -SubAlg(B_1), we have*

$$\dim[B]_{B_1} = \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B' \cap B_1) + \dim \mathbb{U}(C(B)) - \dim \mathbb{U}(B)$$

Now we focus our efforts on $Y(B_2; B)$.

Proposition 4.7. *Assume $Y(B_2; B) \neq \emptyset$. Then $Y(B_2; B)$ is a finite disjoint union of embedded submanifolds of $\mathbb{U}(M_N)$. For each one of these submanifolds there is $u \in Y(B_2; B)$ such that the submanifold's dimension is*

$$\dim \text{Stab}(M_N, B) + \dim \mathbb{U}(B_2) - \dim \text{Stab}(B_2, u^*Bu).$$

Using Proposition 4.5 the later equals

$$\dim \mathbb{U}(B') + \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^*B'u). \quad (2)$$

Proof. We'll define an action on $Y(B_2; B)$ which will partition $Y(B_2; B)$ into a finite number of orbits, each orbit an embedded submanifold of dimension (2) for a corresponding unitary. Define an action of $\text{Stab}(M_N, B) \times \mathbb{U}(B_2)$ on $Y(B_2; B)$ via

$$(w, v).u = wuv^*.$$

For $u \in Y(B_2; B)$ let $\mathcal{O}(u)$ denote the orbit of u and let \mathcal{O} denote the set of all orbits. To prove \mathcal{O} is finite consider the function

$$\begin{aligned} \varphi : \mathcal{O} &\rightarrow \text{*}\text{-SubAlg}(B_2)/\sim_{B_2}, \\ \varphi(\mathcal{O}(u)) &= [u^*Bu]_{B_2}. \end{aligned}$$

Firstly, we need to show φ is well defined. Assume $u_2 \in \mathcal{O}(u_1)$ and take $(w, v) \in \text{Stab}(M_N, B) \times \mathbb{U}(B_2)$ such that $u_2 = wu_1v^*$. From the identities

$$u_2^*Bu_2 = vu_1w^*Bwu_1v^* = vu_1Bu_1v^*$$

we obtain $[u_2Bu_2^*]_{B_2} = [u_1Bu_1^*]_{B_2}$. Hence φ is well defined.

The next step is to show φ is injective. Assume $\varphi(\mathcal{O}(u_1)) = \varphi(\mathcal{O}(u_2))$, for $u_1, u_2 \in Y(B_2; B)$. Since $[u_1^*Bu_1]_{B_2} = [u_2^*Bu_2]_{B_2}$, we have $u_2^*Bu_2 = vu_1^*Bu_1v^*$ for some $v \in \mathbb{U}(B_2)$. But this implies $u_1v^*u_2^* \in \text{Stab}(M_N, B)$ so if $w = u_1v^*u_2^*$ we conclude $(w, v).u_2 = u_1$ which yields $\mathcal{O}(u_1) = \mathcal{O}(u_2)$. We conclude $|\mathcal{O}| \leq |\text{*}\text{-SubAlg}(B_2)/\sim_{B_2}| < \infty$.

Now we prove each orbit is an embedded submanifold of $\mathbb{U}(M_N)$ of dimension (2). Since $\text{Stab}(M_N, B) \times \mathbb{U}(B_2)$ is compact, every orbit $\mathcal{O}(u)$ is compact. Thus, Proposition 3.3 implies $\mathcal{O}(u)$ is an embedded submanifold of $\mathbb{U}(M_N)$, diffeomorphic to

$$(\text{Stab}(M_N, B) \times \mathbb{U}(B_2))/\text{Stab}(u)$$

where

$$\text{Stab}(u) = \{(w, v) \in \text{Stab}(M_N, B) \times \mathbb{U}(B_2) : (w, v).u = u\}.$$

Since

$$(w, v).u = u \iff wuv^* = u \iff u^*wu = v,$$

we deduce the group $\text{Stab}(u)$ is isomorphic to

$$\mathbb{U}(B_2) \cap [u^*\text{Stab}(M_N, B)u],$$

via the map $(w, v) \mapsto v$. A straightforward computation shows

$$u^*\text{Stab}(M_N, B)u = \text{Stab}(M_N, u^*Bu),$$

for any $u \in \mathbb{U}(M_N)$. Hence, for any $u \in Y(B_2; B)$,

$$\dim \mathcal{O}(u) = \dim \text{Stab}(M_N, B) + \mathbb{U}(B_2) - \dim \mathbb{U}(B_2) \cap \text{Stab}(M_N, u^*Bu).$$

Lastly, one can check $\mathbb{U}(B_2) \cap \text{Stab}(M_N, u^*Bu) = \text{Stab}(B_2, u^*Bu)$. \square

Notation 4.8. For a unital C^* -subalgebra B of B_1 , with the property that B is unitarily equivalent to a C^* -subalgebra of B_2 , or in other words $Y(B_2; B)$ is nonempty, define

$$d(B) := \dim[B]_{B_1} + \max_i \{\dim Y_i(B_2; B)\},$$

where $Y_1(B_2, B), \dots, Y_r(B_2; B)$ are disjoint submanifolds of $\mathbb{U}(M_N)$ whose union is $Y(B_2; B)$.

As we mention at the beginning of this section, in order to prove Theorem 4.1, we need to parametrize each $Z(B_1, B_2; [B]_{B_1})$ with a number of coordinates less than N^2 . The number of coordinates will be given by $d(B)$. Thus the next step is to show that, under the hypothesis of Theorem 4.1, we have $d(B) < N^2$ for $B \neq \mathbb{C}$. We will later see that it suffices to show $d(B) < N^2$ for $B \neq \mathbb{C}$ and B abelian.

Before we proceed, we recall definition of multiplicity of a representation. The following lemma combines Lemma III.2.1 in [5] and Theorem 11.9 in [14].

Lemma 4.9. *Suppose $\varphi : A_1 \rightarrow A_2$ is a unital $*$ -homomorphism and A_i is isomorphic to $\bigoplus_{j=1}^{l_i} M_{k_i(j)}$, ($i = 1, 2$). Then φ is determined, up to unitary equivalence in A_2 , by an $l_2 \times l_1$ matrix, written $\mu = \mu(\varphi) = \mu(A_2, A_1)$, having nonnegative integer entries such that*

$$\mu \begin{bmatrix} k_1(1) \\ \vdots \\ k_1(l_1) \end{bmatrix} = \begin{bmatrix} k_2(1) \\ \vdots \\ k_2(l_2) \end{bmatrix}.$$

We call this the matrix of partial multiplicities. In the special case when φ is a unital $$ -representation of A_1 into M_N , μ is a row vector and this vector is called the multiplicity of the representation. One constructs μ as follows: decompose A_p as*

$$A_p = \bigoplus_{j=1}^{l_p} A_p(j)$$

where each $A_p(j)$ is simple, $p = 1, 2$, $1 \leq j \leq l_p$. Taking projections, π induces unital $$ -representations $\pi_i : A_1 \rightarrow A_2(i)$, $1 \leq i \leq l_2$. But up to unitary equivalence, π_i equals*

$$\underbrace{\text{id}_{A_1(1)} \oplus \dots \oplus \text{id}_{A_1(1)}}_{m_{i,1}\text{-times}} \oplus \dots \oplus \underbrace{\text{id}_{A_1(l_1)} \oplus \dots \oplus \text{id}_{A_1(l_1)}}_{m_{i,l_1}\text{-times}}$$

for some nonnegative integer $m_{i,j}$, $1 \leq j \leq l_1$. Set $\mu[i, j] := m_{i,j}$. In particular, $\mu[i, j]$ equals the rank of $\pi_i(p) \in A_2(i)$, where p is a minimal projection in $A_1(j)$. Clearly, π is injective if and only if for all j there is i such that $\mu[i, j] \neq 0$.

Furthermore, the C^* -subalgebra

$$A_2 \cap \varphi(A_1)' = \{x \in A_2 : x\varphi(a) = \varphi(a)x \text{ for all } a \in A_1\}$$

is $*$ -isomorphic to $\bigoplus_{i=1}^{l_2} \bigoplus_{j=1}^{l_1} M_{\mu[i,j]}$ and if we have morphisms $A_1 \rightarrow A_2 \rightarrow A_3$, then $\mu(A_3, A_2)\mu(A_2, A_1) = \mu(A_3, A_1)$ for the corresponding matrices.

Our next task is to show $d(B) < N^2$, for abelian $B \neq C$. We prove it by cases, so let us start.

Lemma 4.10. *Assume B_i is $*$ -isomorphic to M_{k_i} , ($i = 1, 2$) and let $k = \gcd(k_1, k_2)$. Take B a unital C^* -subalgebra of B_1 such that it is unitarily equivalent to a C^* -subalgebra of B_2 . Then there is an injective unital $*$ -representation of B into M_k .*

Proof. Take u in $Y(B_2; B)$ so that $u^*Bu \subseteq B_2$. Let $m_i := \mu(M_N, B_i)$, so that $m_i k_i = N$, ($i = 1, 2$). Find positive integers p_1 and p_2 such that $k_1 = kp_1$ and $k_2 = kp_2$. Assume B is $*$ -isomorphic to $\bigoplus_{j=1}^l M_{n_j}$. To prove the result it is enough to show there are positive integers $m(1), \dots, m(l)$ such that

$$n_1 m(1) + \dots + n_l m(l) = k.$$

Let

$$\begin{aligned} \mu(B_1, B) &= [m_1(1), \dots, m_1(l)], \\ \mu(B_2, u^*Bu) &= [m_2(1), \dots, m_2(l)]. \end{aligned}$$

Since $\mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^*Bu)$ we deduce that $m_1 m_1(j) = m_2 m_2(j)$ for all $1 \leq j \leq l$. Multiplying by k and using $N = m_1 k_1 = m_2 k_2$ we conclude

$$\frac{N}{p_1} m_1(j) = k m_1 m_1(j) = k m_2 m_2(j) = \frac{N}{p_2} m_2(j),$$

so $p_2 m_1(j) = p_1 m_2(j)$. Since $\gcd(p_1, p_2) = 1$, the number $\frac{m_1(j)}{p_1} = \frac{m_2(j)}{p_2}$ is a positive integer whose value we name $m(j)$. From

$$kp_1 = k_1 = \sum_{j=1}^l n_j m_1(j) = \sum_{j=1}^l n_j m(j) p_1,$$

we conclude $k = \sum_{j=1}^l n_j m(j)$. □

Proposition 4.11. *Assume B_1 and B_2 are simple. Take $B \neq \mathbb{C}$ an abelian unital C^* -subalgebra of B_1 , that is unitarily equivalent to a C^* -subalgebra of B_2 . Then $d(B) < N^2$.*

Proof. Assume B_i is $*$ -isomorphic to M_{k_i} , ($i = 1, 2$) and B is $*$ -isomorphic to \mathbb{C}^l , $l \geq 2$. Using Corollary 4.6 and Proposition 4.7, we may take u in $Y(B_2, B)$ such that $d(B)$ equals the sum of the following terms,

$$\begin{aligned} S_1(B) &:= \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap B'), \\ S_2(B) &:= \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u), \\ S_3(B) &:= \dim \mathbb{U}(B'), \end{aligned}$$

Let $k = \gcd(k_1, k_2)$ and write $k_1 = kp_1$, $k_2 = kp_2$. From proof of Lemma 4.10, there are positive integers $m(j)$, $1 \leq j \leq l$, such that

$$\begin{aligned} \mu(B_1, B) &= [m(1)p_1, \dots, m(l)p_1] \\ \mu(B_2, B) &= [m(1)p_2, \dots, m(l)p_2]. \end{aligned}$$

Hence

$$\begin{aligned} S_1(B) &= k_1^2 - \sum_{i=1}^l m(i)^2 p_1^2 = k^2 p_1^2 - \sum_{i=1}^l m(i)^2 p_1^2 \\ S_2(B) &= k_2^2 - \sum_{i=1}^l m(i)^2 p_2^2 = k^2 p_2^2 - \sum_{i=1}^l m(i)^2 p_2^2. \end{aligned}$$

Let $m_i = \mu(M_N, B_i)$, ($i = 1, 2$). Since

$$\mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^* B u),$$

we get

$$\begin{aligned} \mu(M_N, B) &= [m_1 p_1 m(1), \dots, m_1 p_1 m(l)] \\ &= [m_2 p_2 m(1), \dots, m_2 p_2 m(l)]. \end{aligned} \tag{3}$$

Hence

$$S_3(B) = \sum_{i=1}^l (m(i)p_1 m_1)(m(i)p_2 m_2) = \left(\sum_{i=1}^l m(i)^2 \right) p_1 p_2 m_1 m_2.$$

Factoring the term $\sum_{i=1}^l m(i)^2$ we get $d(B)$ equals

$$\left(\sum_{i=1}^l m(i)^2 \right) (p_1 p_2 m_1 m_2 - p_1^2 - p_2^2) + k^2 (p_1^2 + p_2^2).$$

On the other hand, using $N = m_1 k_1 = m_1 k p_1 = m_2 k_2 = m_2 k p_2$, we get $N^2 = k^2 p_1 p_2 m_1 m_2$. Hence $d(B) < N^2$ if and only if

$$\left(\sum_{i=1}^l m(i)^2 \right) (p_1 p_2 m_1 m_2 - p_1^2 - p_2^2) < k^2 (p_1 p_2 m_1 m_2 - p_1^2 - p_2^2). \tag{4}$$

We want to cancel $(p_1 p_2 m_1 m_2 - p_1^2 - p_2^2)$, in equation (4), so we prove it is positive. First we divide it by $p_1 p_2$ to get $m_1 m_2 - \frac{p_1}{p_2} - \frac{p_2}{p_1}$. But from equation (3) we have $\frac{p_1}{p_2} = \frac{m_2}{m_1}$. Thus we need to show $m_1 m_2 - \frac{m_1}{m_2} - \frac{m_2}{m_1}$ is positive. If we divide it by $m_1 m_2$ we get $1 - \frac{1}{m_1^2} - \frac{1}{m_2^2}$, which is clearly positive (recall that $m_1 \geq 2$ and $m_2 \geq 2$ since $B_1 \neq M_N$ and $B_2 \neq M_N$). Therefore, equation (4) is equivalent to

$$\sum_{i=1}^l m(i)^2 < k^2.$$

But $\sum_{i=1}^l m(i) = k$, $l \geq 2$ and each $m(i)$ is positive. \square

In the nonsimple case in Theorem 4.1, we will need some minimization lemmas to show $d(B) < N^2$, for abelian $B \neq \mathbb{C}$. A straightforward use of Lagrange multipliers proves the following lemma, and the one after that is even more elementary.

Lemma 4.12. *Fix a positive integer n and let r_1, \dots, r_n be positive real numbers. Then*

$$\min \left\{ \sum_{j=1}^n \frac{x_j^2}{r_j} \mid \sum_{j=1}^n x_j = 1 \right\} = \frac{1}{\sum_{j=1}^n r_j},$$

where the minimum is taken over all n -tuples of real numbers that sum up to 1.

Lemma 4.13. *For an integer $k \geq 2$ define*

$$h(x, y) = 2xy - \left(1 + \frac{1}{k^2}\right)y^2 - \frac{1}{2}x^2.$$

Then

$$\max\{h(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1/2\} = \frac{1}{4} - \frac{1}{4k^2}.$$

Proposition 4.14. *Suppose $\dim C(B_1) \geq 2$ and B_1 is $*$ -isomorphic to*

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)}. \quad (5)$$

Assume one of the following cases holds:

- (1) $\dim C(B_2) = 1$,
- (2) B_1 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/2},$$

B_2 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/(2k)}$$

where $k \geq 2$.

(3) $\dim C(B_2) \geq 3$ and B_2 is $*$ -isomorphic to

$$M_{N/\dim C(B_2)} \oplus \cdots \oplus M_{N/\dim C(B_2)}.$$

Then for any $B \neq \mathbb{C}$ an abelian unital C^* -subalgebra of B_1 that is unitarily equivalent to a C^* -subalgebra of B_2 , we have that $d(B) < N^2$.

Proof. Let $l_i = \dim C(B_i)$, ($i = 1, 2$), $l = \dim(B)$. Take u in $Y(B_2; B)$ such that $d(B)$ is the sum of the following terms:

$$S_1(B) := \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap B'), \quad (6)$$

$$S_2(B) := \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u), \quad (7)$$

$$S_3(B) := \dim \mathbb{U}(B'). \quad (8)$$

Write

$$\begin{aligned} \mu(B_1, B) &= [a_{i,j}]_{1 \leq i \leq l_1, 1 \leq j \leq l}, \\ \mu(B_2, u^* B u) &= [b_{i,j}]_{1 \leq i \leq l_2, 1 \leq j \leq l}, \\ \mu(M_N, B_1) &= [m_1(1), \dots, m_1(l_1)], \\ \mu(M_N, B_2) &= [m_2(1), \dots, m_2(l_2)], \\ \mu(M_N, B) &= [m(1), \dots, m(l)]. \end{aligned}$$

Then

$$\begin{aligned} S_1(B) &= \frac{N^2}{l_1} - \sum_{i=1}^l \sum_{j=1}^{l_1} a_{i,j}^2, \\ S_2(B) &= \dim \mathbb{U}(B_2) - \sum_{i=1}^l \sum_{j=1}^{l_2} b_{i,j}^2, \\ S_3(B) &= \sum_{j=1}^l m(j)^2. \end{aligned}$$

Since the sum of the ranks appearing in (5) is N , we have $m_1(i) = 1$ for all $1 \leq i \leq l_1$. Since

$$\mu(M_N, B) = \mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^* B u),$$

we must have

$$m(j) = \sum_{i=1}^{l_1} a_{i,j} = \sum_{i=1}^{l_2} m_2(i) b_{i,j}$$

for all $1 \leq j \leq l$. Hence there are nonnegative numbers $\alpha_{i,j}$ and $\beta_{i,j}$ such that $\sum_{i=1}^{l_1} \alpha_{i,j} = \sum_{i=1}^{l_2} \beta_{i,j} = 1$ and $a_{i,j} = \alpha_{i,j} m(j)$, $m_2(i) b_{i,j} =$

$\beta_{i,j}m(j)$. On the other hand, since B is a unital C^* -subalgebra of M_N we must have

$$\sum_{j=1}^l m(j) = N.$$

Thus, there are positive numbers γ_j , ($1 \leq j \leq l$), such that $\sum_{j=1}^l \gamma_j = 1$ and $m(j) = \gamma_j N$. It will be important to notice that $\gamma_j > 0$ for all $1 \leq j \leq l$ (otherwise B is not a unital C^* -algebra of M_N). In consequence,

$$\begin{aligned} S_1(B) &= \frac{N^2}{l_1} - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\sum_{i=1}^{l_1} \alpha_{i,j}^2 \right) \right), \\ S_2(B) &= \dim \mathbb{U}(B_2) - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\sum_{i=1}^{l_2} \frac{\beta_{i,j}^2}{m_2(i)^2} \right) \right), \\ S_3(B) &= N^2 \left(\sum_{j=1}^l \gamma_j^2 \right). \end{aligned}$$

Case (1). B_2 is simple, let us say it is $*$ -isomorphic to M_{k_2} . In this case $\mu(M_N, B_2) = [m_2]$ is just one number and we must have $m_2 k_2 = N$. Notice that $m_2 \geq 2$, since by our standing assumption, $B_2 \neq M_N$. Also notice that from $\mu(M_N, B_2)\mu(B_2, u^*Bu) = \mu(M_N, B)$ we obtain $m_2 b_{i,1} = m(i)$ and $\beta_{i,1} = 1$ for all $1 \leq i \leq l$. In consequence

$$\begin{aligned} S_1(B) &= \frac{N^2}{l_1} - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\sum_{i=1}^{l_1} \alpha_{i,j}^2 \right) \right), \\ S_2(B) &= \frac{N^2}{m_2^2} - \frac{N^2}{m_2^2} \left(\sum_{j=1}^l \gamma_j^2 \right), \\ S_3(B) &= N^2 \left(\sum_{j=1}^l \gamma_j^2 \right). \end{aligned}$$

From Lemma 4.12, we deduce

$$S_1(B) \leq \frac{N^2}{l_1} - \frac{N^2}{l_1} \left(\sum_{j=1}^l \gamma_j^2 \right).$$

Thus, it suffices to show

$$N^2 \left(\frac{1}{l_1} + \frac{1}{m_2^2} + \sum_{j=1}^l \gamma_j^2 - \frac{1}{l_1} \left(\sum_{j=1}^l \gamma_j^2 \right) - \frac{1}{m_2^2} \left(\sum_{j=1}^l \gamma_j^2 \right) \right) < N^2$$

or equivalently

$$\left(\sum_{j=1}^l \gamma_j^2\right) \left(1 - \frac{1}{l_1} - \frac{1}{m_2^2}\right) < 1 - \frac{1}{l_1} - \frac{1}{m_2^2}.$$

Since $l_1 \geq 2$ and $m_2 \geq 2$ we can cancel the term $1 - \frac{1}{l_1} - \frac{1}{m_2^2}$. Thus we need to show $\sum_{j=1}^l \gamma_j^2 < 1$. But the latter follows from the fact that $l \geq 2$, each γ_j is positive and $\sum_{j=1}^l \gamma_j = 1$.

Case (2). We have

$$\begin{aligned} \mu(M_N, B_1) &= [1, 1], \\ \mu(M_N, B_2) &= [1, k]. \end{aligned}$$

Thus

$$\begin{aligned} S_1(B) &= \frac{N^2}{2} - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\alpha_{1,j}^2 + \alpha_{2,j}^2 \right) \right), \\ S_2(B) &= \frac{N^2}{4} + \frac{N^2}{4k^2} - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\beta_{1,j}^2 + \frac{\beta_{2,j}^2}{k^2} \right) \right), \\ S_3(B) &= N^2 \left(\sum_{j=1}^l \gamma_j^2 \right). \end{aligned}$$

From Lemma 4.12 we obtain

$$S_1(B) \leq \frac{N^2}{2} - \frac{N^2}{2} \left(\sum_{j=1}^l \gamma_j^2 \right).$$

Thus, it suffices to show

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4k^2} + \sum_{j=1}^l \gamma_j^2 \left(\frac{1}{2} - \beta_{1,j}^2 - \frac{1}{k^2} \beta_{2,j}^2 \right) < 1$$

or, equivalently,

$$\sum_{j=1}^l \gamma_j^2 \left(\frac{1}{2} - \beta_{1,j}^2 - \frac{1}{k^2} \beta_{2,j}^2 \right) < \frac{1}{4} - \frac{1}{4k^2}.$$

Define

$$r = \sum_{j=1}^l \gamma_j^2 \left(\frac{1}{2} - \beta_{1,j}^2 - \frac{1}{k^2} \beta_{2,j}^2 \right). \quad (9)$$

Now we use the constraints on the variables γ_j and $\beta_{i,j}$. First of all we have $\beta_{1,j} + \beta_{2,j} = 1$ for all $1 \leq i \leq l$. Thus, r simplifies to

$$r = \sum_{j=1}^l \gamma_j^2 \left(2\beta_{2,j} - \left(1 + \frac{1}{k^2} \right) \beta_{2,j}^2 - \frac{1}{2} \right).$$

We also have

$$\sum_{j=1}^l \beta_{2,j} \gamma_j = \frac{1}{2}. \quad (10)$$

Indeed, since all blocks of B are one dimensional, we must have

$$\sum_{j=1}^l b_{2,j} = \frac{N}{2k}.$$

But $kb_{2,j} = \beta_{2,j}m(j) = \beta_{2,j}\gamma_j N$, which implies (10). The final constraint is $\sum_{j=1}^l \gamma_j = 1$.

Now we make the change of variables $q_j := \gamma_j \beta_{2,j}$ and r becomes

$$r = 2 \left(\sum_{j=1}^l q_j \gamma_j \right) - \left(1 + \frac{1}{k^2} \right) \left(\sum_{j=1}^l q_j^2 \right) - \frac{1}{2} \left(\sum_{j=1}^l \gamma_j^2 \right).$$

Letting $\gamma = (\gamma_1, \dots, \gamma_l)$ and $q = (q_1, \dots, q_l)$ and using the Cauchy-Schwartz inequality, we get

$$r \leq 2\|q\|_2\|\gamma\|_2 - \left(1 + \frac{1}{k^2} \right) \|q\|_2^2 - \frac{1}{2} \|\gamma\|_2^2$$

Set $x = \|\gamma\|$, $y = \|q\|$. Notice that $0 \leq x \leq 1$ and $0 \leq y \leq 1/2$. Take

$$h(x, y) = 2xy - \left(1 + \frac{1}{k^2} \right) y^2 - \frac{1}{2} x^2$$

apply Lemma 4.13 to get

$$r \leq h(\|\gamma\|, \|q\|) \leq \frac{1}{4} - \frac{1}{4k^2}.$$

Now we will rule out equality. Assuming, for contradiction, $r = \frac{1}{4} - \frac{1}{4k^2}$, we must have equality in the instance of the Cauchy-Schwartz inequality. Hence $q = z\gamma$ for some real number z . Summing over the coordinates we deduce $z = 1/2$ and then, for all $1 \leq j \leq l$,

$$\frac{1}{2} \gamma_j = q_j = \gamma_j \beta_{2,j}.$$

Since $\gamma_j > 0$ we can cancel and get $\beta_{2,j} = 1/2$. Thus, using the original formulation (9) of r , we get

$$r = \left(\frac{1}{4} - \frac{1}{4k^2} \right) \left(\sum_{j=1}^l \gamma_j^2 \right)$$

which is strictly less than $1/4 - 1/(4k^2)$, because $k \geq 2$, $l \geq 2$, all γ_j are strictly positive and $\sum_{j=1}^l \gamma_j = 1$.

Case (3). Then B_2 is $*$ -isomorphic to

$$\underbrace{M_{N/l_2} \oplus \cdots \oplus M_{N/l_2}}_{l_2\text{-times}}.$$

Arguing as we did before for $m_1(i)$, we have $m_2(i) = 1$, for all $1 \leq i \leq l_2$. Hence

$$\begin{aligned} S_1(B) &= \frac{N^2}{l_1} - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\sum_{i=1}^{l_1} \alpha_{i,j}^2 \right) \right), \\ S_2(B) &= \frac{N^2}{l_2} - N^2 \left(\sum_{j=1}^l \gamma_j^2 \left(\sum_{i=1}^{l_2} \beta_{i,j}^2 \right) \right), \\ S_3(B) &= N^2 \left(\sum_{j=1}^l \gamma_j^2 \right). \end{aligned}$$

From Lemma 4.12 we deduce

$$\begin{aligned} S_1(B) &\leq \frac{N^2}{l_1} - \frac{N^2}{l_1} \left(\sum_{j=1}^l \gamma_j^2 \right), \\ S_2(B) &\leq \frac{N^2}{l_2} - \frac{N^2}{l_2} \left(\sum_{j=1}^l \gamma_j^2 \right). \end{aligned}$$

Thus, it suffices to show

$$N^2 \left(\frac{1}{l_1} + \frac{1}{l_2} + \sum_{j=1}^l \gamma_j^2 - \frac{1}{l_1} \left(\sum_{j=1}^l \gamma_j^2 \right) - \frac{1}{l_2} \left(\sum_{j=1}^l \gamma_j^2 \right) \right) < N^2$$

or equivalently

$$\left(\sum_{j=1}^l \gamma_j^2 \right) \left(1 - \frac{1}{l_1} - \frac{1}{l_2} \right) < 1 - \frac{1}{l_1} - \frac{1}{l_2}.$$

Since $l_1 \geq 2$ and $l_2 \geq 3$ we can cancel the term $1 - \frac{1}{l_1} - \frac{1}{l_2}$ in the above equation and finish the proof as in the previous case. \square

The next step is to find parameterizations of $Z(B_1, B_2; [B]_{B_1})$.

Lemma 4.15. *Take $B \neq \mathbb{C}$ a unital C^* -subalgebra of B_1 that is unitarily equivalent to a C^* -subalgebra of B_2 . If $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) \leq N^2$, B is simple and C in $*\text{-SubAlg}(B)$ is $*$ -isomorphic to \mathbb{C}^2 , then $d(B) \leq d(C)$.*

Proof. Assume B is $*$ -isomorphic to M_k and let m denote the multiplicity of B in M_N . Thus we must have $km = N$. Take a unitary u in the submanifold of maximum dimension in $Y(B_2; B)$, so that $d(B)$ is the sum of the terms

$$\begin{aligned} S_1(B) &:= \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap B'), \\ S_2(B) &:= \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u), \\ S_3(B) &:= \dim \mathbb{U}(B'), \\ S_4(B) &:= \dim \mathbb{U}(B \cap B') - \dim \mathbb{U}(B). \end{aligned}$$

and let v lie in the submanifold of maximum dimension in $Y(B_2, C)$ so that $d(C)$ is the sum of the terms

$$\begin{aligned} S_1(C) &:= \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap C'), \\ S_2(C) &:= \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap v^* C' v), \\ S_3(C) &:= \dim \mathbb{U}(C'). \end{aligned}$$

Clearly, $S_4(B) = 1 - k^2$. We write

$$\begin{aligned} B_1 &\simeq \bigoplus_{i=1}^{l_1} M_{k_1(i)}, \\ B_2 &\simeq \bigoplus_{i=1}^{l_2} M_{k_2(i)}. \end{aligned}$$

and

$$\begin{aligned} \delta(B_1) &= [k_1(1), \dots, k_1(l_1)]^t, \\ \delta(B_2) &= [k_2(1), \dots, k_2(l_2)]^t. \end{aligned}$$

From definition of multiplicity and the fact that it is invariant under unitary equivalence we get

$$\begin{aligned} \mu(B_1, B)k &= \delta(B_1), \\ \mu(B_2, u^* B u)k &= \delta(B_2), \\ \mu(M_N, B_1)\delta(B_1) &= \mu(M_N, B_2)\delta(B_2) = N, \\ \mu(M_N, B_1)\mu(B_1, B) &= \mu(M_N, B_2)\mu(B_2, u^* B u) = m. \end{aligned} \tag{11}$$

From Lemma 4.9 and equation (11) we get

$$\dim \mathbb{U}(B_1 \cap B') = \frac{1}{k^2} \dim \mathbb{U}(B_1). \quad (12)$$

Hence

$$S_1(B) = \left(1 - \frac{1}{k^2}\right) \dim \mathbb{U}(B_1).$$

Similarly

$$S_2(B) = \left(1 - \frac{1}{k^2}\right) \dim \mathbb{U}(B_2).$$

Now it is the turn of C . To ease notation let

$$\mu(B, C) = [x_1, x_2]$$

Notice that $x_1 + x_2 = k$. We claim

$$S_1(C) = \left(1 - \frac{x_1^2 + x_2^2}{k^2}\right) \dim \mathbb{U}(B_1).$$

Using $\mu(B_1, C) = \mu(B_1, B)\mu(B, C)$ we get

$$\dim \mathbb{U}(B_1 \cap C') = (x_1^2 + x_2^2) \dim \mathbb{U}(B_1 \cap B').$$

Furthermore using (12) we obtain

$$\dim \mathbb{U}(B_1 \cap C') = \frac{x_1^2 + x_2^2}{k^2} \dim \mathbb{U}(B_1).$$

Hence our claim follows from definition of $S_1(C)$. Similarly

$$S_2(C) = \left(1 - \frac{x_1^2 + x_2^2}{k^2}\right) \dim \mathbb{U}(B_2).$$

Lastly from $\mu(M_N, C) = [mx_1, mx_2]$ and $mk = N$ we get

$$\begin{aligned} S_3(C) &= (x_1^2 + x_2^2) \frac{N^2}{k^2}, \\ S_3(B) &= \frac{N^2}{k^2}. \end{aligned}$$

To prove $d(B) \leq d(C)$ we'll show

$$S_1(B) - S_1(C) + S_2(B) - S_2(C) + S_4(B) \leq S_3(C) - S_3(B). \quad (13)$$

Using the description of each summand we have that left hand side of (13) equals

$$\frac{x_1^2 + x_2^2 - 1}{k^2} \left(\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) \right) + 1 - k^2.$$

The right hand side of (13) equals

$$\frac{x_1^2 + x_2^2 - 1}{k^2} N^2.$$

But x_1 and x_2 are strictly positive, because C is a unital subalgebra of B . Hence we can cancel $x_1^2 + x_2^2 - 1$ and finish the proof by using that $1 - \delta(B)^2 < 0$ and the assumption $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) \leq N^2$. \square

We recall an important perturbation result that can be found in Lemma III.3.2 from [5].

Lemma 4.16. *Let A be a finite dimensional C^* -algebra. Given any positive number ε there is a positive number $\delta = \delta(\varepsilon)$ so that whenever B and C are unital C^* -subalgebras of A and such that C has a system of matrix units $\{e_C(s, i, j)\}_{s,i,j}$, satisfying $\text{dist}(e_C(s, i, j), B) < \delta$ for all s, i and j , then there is a unitary u in $\mathbb{U}(C^*(B, C))$ with $\|u - 1\| < \varepsilon$ so that $uCu^* \subseteq B$.*

Notation 4.17. For an element x in M_N and a positive number ε , $\mathcal{N}_\varepsilon(x)$ denotes the open ε -neighborhood around x (i.e. open ball of radius ε centered at x), where the distance is from the operator norm in M_N .

The next proposition is quite technical and is mainly a consequence of Lemma 4.16. The set $[B]_{B_1}$ is endowed with the equivalent topologies described in Remark 4.3.

Lemma 4.18. *Take B in $\ast\text{-SubAlg}(B_1)$ and assume $Z(B_1, B_2; [B]_{B_1})$ is nonempty. Then the function*

$$\begin{aligned} Z(B_1, B_2; [B]_{B_1}) &\rightarrow [B]_{B_1} \\ u &\mapsto uB_2u^* \cap B_1 \end{aligned} \tag{14}$$

is continuous.

Proof. Assume B is \ast -isomorphic to

$$\bigoplus_{s=1}^l M_{k_s}.$$

First we recall that the topology of $[B]_{B_1}$ is induced by the bijection

$$\begin{aligned} \beta : [B]_{B_1} &\rightarrow \mathbb{U}(B_1)/\text{Stab}(B_1, B), \\ \beta(uBu^*) &= u\text{Stab}(B_1, B). \end{aligned}$$

For convenience let $\pi : \mathbb{U}(B_1) \rightarrow \mathbb{U}(B_1)/\text{Stab}(B_1, B)$ denote the canonical quotient map. Pick u_0 in $Z(B_1, B_2; [B]_{B_1})$. With no loss of generality we may assume $B = u_0B_2u_0^* \cap B_1$.

We prove the result by contradiction. Suppose the function in (14) is not continuous at u_0 . Then there is a sequence $(u_k)_{k \geq 1} \subset Z(B_1, B_2, [B]_{B_1})$ and an open neighborhood \mathcal{N} of B in $[B]_{B_1}$ such that

- (1) $\lim_k u_k = u_0$,
- (2) for all k , $u_k B_2 u_k^* \cap B_1 \notin \mathcal{N}$.

On the other hand, let $\varepsilon > 0$ be such that $\pi(\mathcal{N}_\varepsilon(1_{B_1})) \subseteq \beta(\mathcal{N})$. Let $\{e_k(s, i, j)\}_{1 \leq s \leq l, 1 \leq i, j \leq k_s}$ denote a system of matrix units for $u_k B_2 u_k^* \cap B_1$. Fix elements $f_k(s, i, j)$ in B_2 such that $e_k(s, i, j) = u_k f_k(s, i, j) u_k^*$. Since B_2 is finite dimensional, passing to a subsequence if necessary, we may assume that $\lim_k f_k(s, i, j) = f(s, i, j)$, for all s, i and j . Using property (1) of the sequence $(u_k)_{k \geq 1}$, we deduce

$$\lim_k e_k(s, i, j) = \lim_k u_k f_k(s, i, j) u_k^* = u_0 f(s, i, j) u_0^*.$$

Hence the element $e(s, i, j) = u_0 f(s, i, j) u_0^*$ belongs to $u_0 B_1 u_0^* \cap B_1 = B$. Use Lemma 4.16 and take δ_1 positive such that whenever C is a subalgebra in $\ast\text{-SubAlg}(B_1)$ having a system of matrix units $\{e_C(s, i, j)\}_{s, i, j}$ satisfying $\text{dist}(e_C(s, i, j), B) < \delta_1$, for all s, i and j , then there is a unitary Q in $\mathbb{U}(B_1)$ such that $\|Q - 1_{B_1}\| < \varepsilon$ and $Q C Q^* \subseteq B$. Take k such that $\|e_k(s, i, j) - e(s, i, j)\| < \delta_1$ for all s, i and j . This implies $\text{dist}(e_k(s, i, j), B) < \delta_1$ for all s, i and j . We conclude there is a unitary Q in $\mathbb{U}(B_1)$ such that $\|Q - 1_{B_1}\| < \varepsilon$ and $Q^*(u_k B_2 u_k^* \cap B_1) Q \subseteq B$. But

$$\dim B = \dim u_k B_2 u_k^* \cap B_1 = \dim Q^*(u_k B_2 u_k^* \cap B_1) Q,$$

where in the first equality we used that u_k lies in $Z(B_1, B_2; [B]_{B_1})$. Hence $Q^*(u_k B_2 u_k^* \cap B_1) Q = B$. As a consequence,

$$\beta(u_k B_2 u_k^* \cap B_1) = \beta(Q B Q^*) = \pi(Q) \in \beta(\mathcal{N}).$$

But the latter contradicts property (2) of $(u_k)_{k \geq 1}$. \square

Lemma 4.19. *For B in $\ast\text{-SubAlg}(B)$, the function $c : [B]_{B_1} \rightarrow [C(B)]_{B_1}$ given by $c(u B u^*) = u C(B) u^*$ is continuous.*

Proof. First, we must show the function c is well defined. In other words we have to show $\text{Stab}(B_1, B) \subseteq \text{Stab}(B_1, C(B))$. But this follows directly from the fact that any u in $\text{Stab}(B_1, B)$ defines a \ast -automorphism of B and any \ast -automorphism leaves the center fixed. Since $[B]_{B_1}$ and $[C(B)]_{B_1}$ are homeomorphic to $\mathbb{U}(B_1)/\text{Stab}(B_1, B)$ and $\mathbb{U}(B_1)/\text{Stab}(B_1, C(B))$ respectively, it follows that c is continuous if and only if the function $\tilde{c} : \mathbb{U}(B_1)/\text{Stab}(B_1, B) \rightarrow \mathbb{U}(B_1)/\text{Stab}(B_1, C(B))$ given by $\tilde{c}(u \text{Stab}(B_1, B)) = u \text{Stab}(B_1, C(B))$ is continuous. But the spaces $\mathbb{U}(B_1)/\text{Stab}(B_1, B)$ and $\mathbb{U}(B_1)/\text{Stab}(B_1, C(B))$ have the quotient topology induced by the canonical projections

$$\pi_B : \mathbb{U}(B_1) \rightarrow \text{Stab}(B_1, B), \quad \pi_{C(B)} : \mathbb{U}(B_1) \rightarrow \mathbb{U}(B_1)/\text{Stab}(B_1, C(B)).$$

Thus \tilde{c} is continuous if and only if $\pi_B \circ \tilde{c}$ is continuous. But $\pi_B \circ \tilde{c} = \pi_{C(B)}$, which is indeed continuous. \square

We are ready to find local parameterizations of $Z(B_1, B_2; [B]_{B_1})$.

Proposition 4.20. *Take B a unital C^* -subalgebra in B_1 that is unitarily equivalent to a C^* -subalgebra of B_2 . Fix an element u_0 in $Z(B_1, B_2; [B]_{B_1})$. Then there is a positive number r and a continuous injective function*

$$\Psi : \mathcal{N}_r(u_0) \cap Z(B_1, B_2; [B]_{B_1}) \rightarrow \mathbb{R}^{d(C(B))}.$$

Proof. Using that $Z(B_1, B_2; [B]_{B_1}) = Z(B_1, B_2, [u_0 B_2 u_0^* \cap B_1]_{B_1})$, with no loss of generality we may assume $u_0 B_2 u_0^* \cap B_1 = B$. Now, we use the manifold structure of $[C(B)]_{B_1}$ and $Y(B_2; C(B))$ to construct Ψ . Note that if $Y(B_2, B)$ is nonempty then $Y(B_2, C(B))$ is nonempty as well. Let d_1 denote the dimension of $[C(B)]_{B_1}$ and let d_2 denote the dimension of the submanifold of $Y(B_2; C(B))$ that contains u_0 . Of course, we have $d_1 + d_2 \leq d(C(B))$.

We use the local cross section result from previous section to parametrize $[C(B)]_{B_1}$. To ease notation take $G = \mathbb{U}(B_1)$, $H = \text{Stab}(B_1, C(B))$ and let π denote the canonical quotient map from G onto the left-cosets of H . By Proposition 3.5 there are

- (1) \mathcal{N}_G , a compact neighborhood of 1 in G ,
- (2) \mathcal{N}_H , a compact neighborhood of 1 in H ,
- (3) $\mathcal{N}_{G/H}$, a compact neighborhood of $\pi(1)$ in G/H ,
- (4) a continuous function $s : \mathcal{N}_{G/H} \rightarrow \mathcal{N}_G$ satisfying
 - (a) $s(\pi(1)) = 1$ and $\pi(s(\pi(g))) = \pi(g)$ whenever $\pi(g)$ lies in $\mathcal{N}_{G/H}$,
 - (b) the function

$$\begin{aligned} \mathcal{N}_H \times \mathcal{N}_{G/H} &\rightarrow \mathcal{N}_G, \\ (h, \pi(g)) &\mapsto hs(\pi(g)), \end{aligned}$$

is an homeomorphism.

Since G/H is a manifold of dimension d_1 , we may assume there is a continuous injective map $\Psi_1 : \mathcal{N}_{G/H} \rightarrow \mathbb{R}^{d_1}$.

Parametrizing $Y(B_2; C(B))$ is easier. Since $u_0 B_2 u_0^* \cap B_1 = B$, u_0 belongs to $Y(B_2; B)$. Take r_1 positive and a diffeomorphism Ψ_2 from $Y(B_2; C(B)) \cap \mathcal{N}_{r_1}(u_0)$ onto an open subset of \mathbb{R}^{d_2} .

Now that we have fixed parametrizations Ψ_1 and Ψ_2 , we can parametrize $Z(B_1, B_2; [B]_{B_1})$ around u_0 . Recall $[C(B)]_{B_1}$ has the topology induced by the bijection $\beta : [C(B)]_{B_1} \rightarrow G/H$, given by $\beta(uC(B)u^*) =$

$\pi(u)$. The function

$$\begin{aligned} Z(B_1, B_2; [B]_{B_1}) &\rightarrow [C(B)]_{B_1}, \\ u &\mapsto c(uB_2u^* \cap B_1) \end{aligned}$$

is continuous by Lemma 4.18 and Lemma 4.19. Hence there is δ_2 positive such that $\beta(c(uB_2u^* \cap B_1))$ belongs to $\mathcal{N}_{G/H}$, whenever u lies in the intersection $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$. For a unitary u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$ define

$$q(u) := s(\beta(c(uB_2u^* \cap B_1))).$$

We note that $q(u_0) = 1$, $q(u)$ lies in G and that the map $u \mapsto q(u)$ is continuous. The main property of $q(u)$ is that

$$c(uB_2u^* \cap B_1) = q(u)c(B)q(u)^*. \quad (15)$$

Indeed, for u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$ there is a unitary v in G with the property $uB_2u^* \cap B_1 = vBv^*$. Hence $c(uB_2u^* \cap B_1) = vC(B)v^*$. Since $\|u - u_0\| < \delta_2$, $\beta(c(uB_2u^* \cap B_1))$ lies in $\mathcal{N}_{G/H}$. Hence $\beta(c(uB_2u^* \cap B_1)) = \pi(v)$ lies in $\mathcal{N}_{G/H}$. Using the fact that s is a local section on $\mathcal{N}_{G/H}$ (property (4a) above) we deduce $\pi(s(\pi(v))) = \pi(v)$.

On the other hand, by definition of $q(u)$ we have

$$\pi(s(\pi(v))) = \pi(s(\beta(c(uB_2u^* \cap B_1)))) = \pi(q(u)).$$

As a consequence, $\pi(v) = \pi(q(u))$ i.e. $v^*q(u)$ belongs to $\text{Stab}(B_1, B)$ which is just another way to say (15) holds. At last we are ready to find r . Continuity of the map $u \mapsto q(u)$ gives a positive δ_3 , less than δ_2 , such that $\|q(u) - 1\| < \frac{\delta_1}{2}$ whenever u lies in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_3}(u_0)$. Define $r = \min\{\frac{\delta_1}{2}, \delta_3\}$. The first thing we notice is that $q(u)^*u$ belongs to $Y(B_2; C(B)) \cap \mathcal{N}_{\delta_1}(u_0)$ whenever u lies in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_\delta(u_0)$. Indeed, from

$$q(u)c(B)q(u)^* = c(uB_2u^* \cap B_1) \subseteq uB_2u^*$$

we obtain $q(u)^*u \in Y(B_2; c(B))$ and a standard computation, using $\|q(u) - 1\| < \frac{\delta_1}{2}$, shows $\|q(u)^*u - u_0\| < \delta_1$. Hence we are allowed to take $\Psi_2(q(u)^*u)$. Lastly, for u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_\delta(u_0)$ define

$$\Psi(u) := (\Psi_1(\beta(c(uB_2u^* \cap B_1))), \Psi_2(q(u)u^*)).$$

It is clear that Ψ is continuous.

Now we show Ψ is injective. If $\Psi(u_1) = \Psi(u_2)$, for two element u_1 and u_2 in $Z(B_1, B_2; [B]_{B_1})$, then

$$\Psi_1(\beta(c(u_1B_2u_1^* \cap B_1))) = \Psi_1(\beta(c(u_2B_2u_2^* \cap B_1))), \quad (16)$$

$$\Psi_2(q(u_1)u_1^*) = \Psi_2(q(u_2)u_2^*). \quad (17)$$

From (16) and definition of $q(u)$ it follows that $q(u_1) = q(u_2)$ and from equation (17) we conclude $u_1 = u_2$. \square

Proposition 4.21. *Take B a unital C^* -subalgebra of B_1 such that it is unitarily equivalent to a C^* -subalgebra of B_2 . Fix an element u_0 in $Z(B_1, B_2; [B]_{B_1})$.*

There is a positive number r and a continuous injective function

$$\Psi : \mathcal{N}_r(u_0) \cap Z(B_1, B_2; [B]_{B_1}) \rightarrow \mathbb{R}^{d(B)}$$

The proof of Proposition 4.21 is similar to that of Proposition 4.20, so we omit it.

We now begin showing density in $\mathbb{U}(M_N)$ of certain sets of unitaries.

Lemma 4.22. *Assume B_1 and B_2 are simple. If $B \neq \mathbb{C}$ is a unital C^* -subalgebra of B_1 and it is unitarily equivalent to a C^* -subalgebra of B_2 then $Z(B_1, B_2; [B]_{B_1})^c$ is dense.*

Proof. Firstly we notice that $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) < N^2$. Indeed, if B_i is $*$ -isomorphic to M_{k_i} , $i = 1, 2$ and $m_i = \mu(M_N, B_i)$ then $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) = N^2(1/m_1^2 + 1/m_2^2) < N^2$. Secondly we will prove that for any u in $Z(B_1, B_2; [B]_{B_1})$ there is a natural number d_u , with $d_u < N^2$, a positive number r_u and a continuous injective function $\Psi_u : \mathcal{N}_{r_u}(u) \cap Z(B_1, B_2; [B]_{B_1}) \rightarrow \mathbb{R}^{d_u}$. We will consider two cases.

Case (1): B is not simple. Take $d_u = d(C(B))$. Since $C(B) \neq \mathbb{C}$, Proposition 4.11 implies $d(C(B)) < N^2$. Take r_u and Ψ_u as required to exist by Proposition 4.20.

Case (2): B is simple. Take $d_u = d(B)$. Since $B \neq \mathbb{C}$, B contains a unital C^* -subalgebra isomorphic to \mathbb{C}^2 , call it C . Lemma 4.15 implies $d(B) \leq d(C)$ and Lemma 4.11 implies $d(C) < N^2$. Take r_u and Ψ_u the positive number and continuous injective function from Proposition 4.21.

We will show that $U \cap Z(B_1, B_2; [B]_{B_1})^c \neq \emptyset$, for any nonempty open subset $U \subseteq \mathbb{U}(M_N)$. First notice that if the intersection $U \cap (\bigcup_{u \in Z(B_1, B_2; [B]_{B_1})} \mathcal{N}_{r_u}(u))^c$ is nonempty then we are done. Thus we may assume $U \subseteq \bigcup_{u \in Z(B_1, B_2; [B]_{B_1})} \mathcal{N}_{r_u}(u)$. Furthermore, by making U smaller, if necessary, we may assume there is u in $Z(B_1, B_2; [B]_{B_1})$ such that $U \subseteq \mathcal{N}_{r_u}(u)$.

For sake of contradiction assume $U \subseteq Z(B_1, B_2; [B]_{B_1})$. We may take an open subset V , contained in U , small enough so that V is diffeomorphic to an open connected set \mathcal{O} of \mathbb{R}^{N^2} . Let $\varphi : \mathcal{O} \rightarrow V$ be

a diffeomorphism. It follows we have a continuous injective function

$$\mathbb{R}^{N^2} \supseteq \mathcal{O} \xrightarrow{\varphi} V \xrightarrow{\Psi_u} \mathbb{R}^{d_u} \hookrightarrow \mathbb{R}^{N^2}.$$

By the Invariance of Domain Theorem, the image of this map must be open in \mathbb{R}^{N^2} . But this is a contradiction since the image is contained in \mathbb{R}^{d_u} and $d_u < N^2$. We conclude $U \cap Z(B_1, B_2; [B]_{B_1})^c \neq \emptyset$. \square

Lemma 4.23. *Suppose $\dim C(B_1) \geq 2$ and B_1 is $*$ -isomorphic to*

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)}.$$

Assume one of the following cases holds:

- (1) $\dim C(B_2) = 1$,
- (2) B_1 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/2}$$

and B_2 is $$ -isomorphic to*

$$M_{N/2} \oplus M_{N/(2k)},$$

where $k \geq 2$.

- (3) $\dim C(B_2) \geq 3$ and B_2 is $*$ -isomorphic to

$$M_{N/\dim C(B_2)} \oplus \cdots \oplus M_{N/\dim C(B_2)}.$$

Then for any $B \neq \mathbb{C}$ unital C^ -subalgebra of B_1 such that it is unitarily equivalent to a C^* -subalgebra of B_2 , $Z(B_1, B_2; [B]_{B_1})^c$ is dense.*

Proof. The proof of Lemma 4.23 is exactly as the proof of 4.22 but using Lemma 4.14 instead of Lemma 4.11. \square

At this point if the sets $Z(B_1, B_2; [B]_{B_1})$ were closed one could conclude immediately that $\Delta(B_1, B_2)$ is dense. Unfortunately they may not be closed. What saves the day is the fact that we can control the closure of $Z(B_1, B_2; [B]_{B_1})$ with sets of the same form i.e. sets like $Z(B_1, B_2; [C]_{B_1})$ for a suitable finite family of subalgebras C . We make this statement clearer with the definition of an order on $*$ -SubAlg(B_1).

Definition 4.24. On $*$ -SubAlg(B_1)/ \sim_{B_1} we define a partial order as follows:

$$[B]_{B_1} \leq [C]_{B_1} \Leftrightarrow \exists D \in \text{*-SubAlg}(C) : D \sim_{B_1} B.$$

Proposition 4.25. *For any B in $*$ -SubAlg(B_1),*

$$\overline{Z(B_1, B_2; [B]_{B_1})} \subseteq \bigcup_{[C]_{B_1} \geq [B]_{B_1}} Z(B_1, B_2; [C]_{B_1}).$$

Proof. Let $(u_k)_{k \geq 1}$ be a sequence in $Z(B_1, B_2; [B]_{B_1})$ and u in $\mathbb{U}(M_N)$ such that $\lim_k \|u_k - u\| = 0$. Pick q_k in $\mathbb{U}(M_N)$ such that $q_k B q_k^* = u_k B_2 u_k^* \cap B_1$. Let $\{f_k(s, i, j)\}_{s, i, j}$ be a matrix unit for $u_k B_2 u_k^* \cap B_1$ and take elements $e_k(s, i, j)$ in B_2 such that $f_k(s, i, j) = u_k e_k(s, i, j) u_k^*$. Since B_2 is finite dimensional, passing to a subsequence if necessary, we may assume $\lim_k f_k(s, i, j) = f(s, i, j) \in B_2$ and $\lim_k u_k e_k(s, i, j) u_k^* = u e(s, i, j) u^*$ for some $e(s, i, j) \in B_1$, for all s, i and j . It follows that $\lim_k \text{dist}(f_k(s, i, j), u B_2 u^* \cap B_1) = 0$. Hence, from Lemma 4.16, for large k , there is q in $\mathbb{U}(M_N)$ so that $q(u_k B_2 u_k^* \cap B_1) q^* = q q_k B q_k^* q^*$ is contained in $u B_2 u^* \cap B_1$. We conclude $[u B_2 u^* \cap B_1]_{B_1} \geq [B]_{B_1}$ and since u lies in $Z(B_1, B_2; [u B_2 u^* \cap B_1])$ the proof is complete. \square

Lemma 4.26. *Assume one of the following conditions holds:*

- (1) $\dim C(B_1) = 1 = \dim C(B_2)$,
- (2) $\dim C(B_1) \geq 2$, $\dim C(B_2) = 1$ and B_1 is $*$ -isomorphic to

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)},$$

- (3) $\dim C(B_1) = 2 = \dim C(B_2)$, B_1 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/2},$$

and B_2 is $*$ -isomorphic to

$$M_{N/2} \oplus M_{N/(2k)}$$

where $k \geq 2$,

- (4) $\dim C(B_1) \geq 2$, $\dim C(B_2) \geq 3$ and, for $i = 1, 2$, B_i is $*$ -isomorphic to

$$M_{N/\dim C(B_i)} \oplus \cdots \oplus M_{N/\dim C(B_i)}.$$

Take B a unital C^* -subalgebra of B_1 such that it is unitarily equivalent to a C^* -subalgebra of B_2 . If $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is not dense and $B \neq \mathbb{C}$ then there is a subalgebra C in $*$ -SubAlg(B_1) such that $[C]_{B_1} > [B]_{B_1}$ and $\overline{Z(B_1, B_2; [C]_{B_1})}^c$ is not dense.

Proof. We proceed by contrapositive. Thus, assume $\overline{Z(B_1, B_2; [C]_{B_1})}^c$ is dense for all $[C]_{B_1} > [B]_{B_1}$. Since the set $\{[C]_{B_1} : [C]_{B_1} > [B]_{B_1}\}$ is finite,

$$\bigcap_{[C]_{B_1} > [B]_{B_1}} \overline{Z(B_1, B_2; [C]_{B_1})}^c$$

is open and dense. Furthermore, Lemma 4.22 or Lemma 4.23 implies $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is dense. Hence the intersection

$$\overline{Z(B_1, B_2; [B]_{B_1})}^c \cap \bigcap_{[C]_{B_1} > [B]_{B_1}} \overline{Z(B_1, B_2; [C]_{B_1})}^c$$

is dense. But this along with Proposition 4.25 implies $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is dense. \square

Lemma 4.27. *Assume one of the conditions (1)–(4) of Lemma 4.26 holds. Then for any $B \neq \mathbb{C}$, unital C^* -subalgebra of B_1 that is unitarily equivalent to a C^* -subalgebra of B_2 , the set $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is dense.*

Proof. Assume $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is not dense. By Lemma 4.26 there is $[C]_{B_1} > [B]_{B_1}$ such that $\overline{Z(B_1, B_2; [C]_{B_1})}^c$ is not dense. We notice that again we are in the same condition to apply Lemma 4.26, since $[C]_{B_1} > [B]_{B_1} > [\mathbb{C}]_{B_1}$. In this way we can construct chains, in $*\text{-SubAlg}(B_1)/\sim_{B_1}$, of length arbitrarily large, but this can not be since it is finite. \square

At last we can give a proof of Theorem 4.1.

Proof of Theorem 4.1. A direct computation shows that

$$\Delta(B_1, B_2) = \bigcap_{[B]_{B_1} > [\mathbb{C}]_{B_1}} Z(B_1, B_2, [B]_{B_1})^c.$$

Thus

$$\Delta(B_1, B_2) \supseteq \bigcap_{[B]_{B_1} > [\mathbb{C}]_{B_1}} \overline{Z(B_1, B_2, [B]_{B_1})}^c.$$

Now, by Lemma 4.27, whenever $[B]_{B_1} > [\mathbb{C}]_{B_1}$, the set $\overline{Z(B_1, B_2, [B]_{B_1})}^c$ is dense. Hence $\Delta(B_1, B_2)$ is dense. \square

5. PRIMITIVITY

During this section, unless stated otherwise, $A_1 \neq \mathbb{C}$ and $A_2 \neq \mathbb{C}$ denote two nontrivial, separable, residually finite dimensional C^* -algebras. Our goal is to prove $A_1 * A_2$ is primitive, except for the case $A_1 = \mathbb{C}^2 = A_2$. Two main ingredients are used. Firstly, the perturbation results from previous section. Secondly, the fact that $A_1 * A_2$ has a separating family of finite dimensional $*$ -representations, a result due to Excel and Loring, [7].

Before we start proving results about primitivity, we want to consider the case $\mathbb{C}^2 * \mathbb{C}^2$. This is a well studied C^* -algebra; see for instance [3], [12] and [13]. It is known that $\mathbb{C}^2 * \mathbb{C}^2$ is $*$ -isomorphic to the C^* -algebra of continuous M_2 -valued functions on the closed interval $[0, 1]$, whose values at 0 and 1 are diagonal matrices. As a consequences its center is not trivial. Since the center of any primitive C^* -algebra is trivial, we conclude $\mathbb{C}^2 * \mathbb{C}^2$ is not primitive.

Definition 5.1. We denote by ι_j the inclusion $*$ -homomorphism from A_j into $A_1 * A_2$. Given a unital $*$ -representation $\pi : A_1 * A_2 \rightarrow \mathbb{B}(H)$, we define $\pi^{(1)} = \pi \circ \iota_1$ and $\pi^{(2)} = \pi \circ \iota_2$. Thus, with this notation, we have $\pi = \pi^{(1)} * \pi^{(2)}$. For a unitary u in $\mathbb{U}(H)$ we call the $*$ -representation $\pi^{(1)} * (\text{Ad } u \circ \pi^{(2)})$, a perturbation of π by u .

Remark 5.2. The $*$ -representation $\pi^{(1)} * (\text{Ad } u \circ \pi^{(2)})$ is irreducible if and only if

$$u\pi^{(2)}(A_2)'u^* \cap \pi^{(1)}(A_1)' = \mathbb{C}.$$

where $(\pi^{(1)}(A_1))'$ denotes de commutant of $\pi^{(1)}(A_1)$ in $\mathbb{B}(H)$.

Our first goal is to perturb a given finite dimensional $*$ -representation of $A_1 * A_2$ into an irreducible one. Of course, the example $\mathbb{C}^2 * \mathbb{C}^2$ shows that in general this can't be done so we have to find conditions that guarantee it. We start with the case A_1 and A_2 finite dimensional and later, built on the finite dimensional case, we continue with the residually finite dimensional case. For the finite dimensional case crucial information is given by the ranks of minimal central projections on A_1 and A_2 .

Definition 5.3. Assume A_1 and A_2 are finite dimensional and let $\rho : A_1 * A_2 \rightarrow \mathbb{B}(H)$ be a unital, finite dimensional representation. We say that ρ satisfies the *Rank of Central Projections condition* (or *RCP condition*) if for both $i = 1, 2$, the rank of $\rho(p)$ is the same for all minimal projections p of the center $C(A_i)$ of A_i , (but they need not agree for different values of i).

The RCP condition for ρ , of course, is really about the pair of representations $(\rho^{(1)}, \rho^{(2)})$. However, it will be convenient to express it in terms of $A_1 * A_2$. In any case, the following two lemmas are clear.

Lemma 5.4. *Suppose A_1 and A_2 are finite dimensional, $\rho : A_1 * A_2 \rightarrow \mathbb{B}(H)$ is a finite dimensional representation that satisfies the RCP condition and $u \in \mathbb{U}(H)$. Then the representation $\rho^{(1)} * (\text{Ad } u \circ \rho^{(2)})$ of $A_1 * A_2$ also satisfies the RCP condition.*

Lemma 5.5. *Suppose A_1 and A_2 are finite dimensional, $\rho : A_1 * A_2 \rightarrow \mathbb{B}(H)$ and $\sigma : A_1 * A_2 \rightarrow \mathbb{B}(K)$ are finite dimensional representations that satisfy the RCP condition. Then $\rho \oplus \sigma : A_1 * A_2 \rightarrow \mathbb{B}(H \oplus K)$ also satisfies the RCP condition.*

The following is clear from Lemma 4.9.

Lemma 5.6. *Assume A is a finite dimensional C^* -algebra $*$ -isomorphic to $\bigoplus_{j=1}^l M_{n(j)}$ and take $\pi : A \rightarrow \mathbb{B}(H)$ a unital finite dimensional $*$ -representation. Let $\mu(\pi) = [m(1), \dots, m(l)]$ and let $\tilde{\pi}$ be the restriction*

of π to the center of A . Then

$$\mu(\tilde{\pi}) = [m(1)n(1), \dots, m(l)n(l)].$$

The next lemma will help us to prove that the RCP condition is easy to get.

Lemma 5.7. *Assume A is a finite dimensional C^* -algebra and $\pi : A \rightarrow \mathbb{B}(H)$ is a unital finite dimensional $*$ -representation. Let*

$$\mu(\pi) = [m(1), \dots, m(l)].$$

For any nonnegative integers $q(1), \dots, q(l)$ there is a finite dimensional unital $$ -representation $\rho : A \rightarrow \mathbb{B}(K)$ such that*

$$\mu(\pi \oplus \rho) = [m(1) + q(1), \dots, m(l) + q(l)].$$

Proof. Write A as

$$A = \bigoplus_{i=1}^l A(i)$$

where $A(i) = \mathbb{B}(V_i)$ for V_i finite dimensional. For $1 \leq i \leq l$, let $p_i : A \rightarrow A(i)$ denote the canonical projection onto $A(i)$. Notice that p_i is a unital $*$ -representation of A . Define

$$\rho := \bigoplus_{i=1}^l \underbrace{(p_i \oplus \dots \oplus p_i)}_{q(i)\text{-times}} : A \rightarrow \bigoplus_{i=1}^l A(i)^{q(i)} \subseteq \mathbb{B}(K),$$

where $K = \bigoplus_{i=1}^l (V_i^{\oplus q_i})$. Then ρ is a unital $*$ -representation of A on K and

$$\mu(\pi \oplus \rho) = [m(1) + q(1), \dots, m(l) + q(l)].$$

□

The next lemma takes slightly more work and is essential to our construction.

Lemma 5.8. *Assume A_1 and A_2 are finite dimensional. Given a unital finite dimensional $*$ -representation $\pi : A_1 * A_2 \rightarrow \mathbb{B}(H)$, there is a finite dimensional Hilbert space \hat{H} and a unital $*$ -representation*

$$\hat{\pi} : A_1 * A_2 \rightarrow \mathbb{B}(\hat{H})$$

such that $\pi \oplus \hat{\pi}$ satisfies the RCP condition.

Proof. For $i = 1, 2$, let $l_i = \dim C(A_i)$, let A_i be $*$ -isomorphic to $\bigoplus_{j=1}^{l_i} M_{n_i(j)}$ and write

$$\mu(\pi^{(i)}) = [m_i(1), \dots, m_i(l_i)].$$

Take $n_i = \text{lcm}(n_i(1), \dots, n_i(l_i))$ and integers $r_i(j)$, such that $r_i(j)n_i(j) = n_i$, for $1 \leq j \leq l_i$. Take a positive integer s such that $sr_i(j) \geq m_i(j)$ for all $i = 1, 2$ and $1 \leq j \leq l_i$. Use Lemma 5.7 to find a unital finite dimensional $*$ -representation $\rho_i : A_i \rightarrow \mathbb{B}(K_i)$, $i = 1, 2$ such that

$$\mu(\pi^{(i)} \oplus \rho_i) = [sr_i(1), \dots, sr_i(l_i)].$$

Letting κ_i denote the restriction of $\pi^{(i)} \oplus \rho_i$ to $C(A_i)$, from Lemma 5.6 we have

$$\mu(\kappa_i) = [sr_i(1)n_i(1), \dots, sr_i(l_i)n_i(l_i)] = [sn_i, sn_i, \dots, sn_i].$$

The $*$ -representations $(\pi^{(1)} \oplus \rho_1)$ and $(\pi^{(2)} \oplus \rho_2)$ are almost what we want, but they may take values in Hilbert spaces with different dimensions. To take care of this, we take multiples of them. Let $N = \text{lcm}(\dim(H \oplus K_1), \dim(H \oplus K_2))$, find positive integers k_1 and k_2 such that

$$N = k_1 \dim(H \oplus K_1) = k_2 \dim(H \oplus K_2)$$

and consider the Hilbert spaces $(H \oplus K_i)^{\oplus k_i}$, whose dimensions agree for $i = 1, 2$. Then

$$\dim(K_1 \oplus (H \oplus K_1)^{\oplus(k_1-1)}) = \dim(K_2 \oplus (H \oplus K_2)^{\oplus(k_2-1)})$$

and there is a unitary operator

$$U : K_2 \oplus (H \oplus K_2)^{\oplus(k_2-1)} \rightarrow K_1 \oplus (H \oplus K_1)^{\oplus(k_1-1)}.$$

Take

$$\begin{aligned} \hat{H} &:= K_1 \oplus (H \oplus K_1)^{\oplus(k_1-1)}, \\ \hat{\pi}_1 &:= \rho_1 \oplus (\pi^{(1)} \oplus \rho)^{\oplus(k_1-1)}, \\ \sigma_1 &:= \pi^{(1)} \oplus \hat{\pi}_1, \\ \hat{\pi}_2 &:= \text{Ad } U \circ (\rho_2 \oplus (\pi^{(2)} \oplus \rho)^{\oplus(k_2-1)}), \\ \sigma_2 &:= \pi^{(2)} \oplus \hat{\pi}_2, \\ \hat{\pi} &:= \hat{\pi}_1 * \hat{\pi}_2. \end{aligned}$$

Then $\sigma_1 * \sigma_2 = (\pi^{(1)} \oplus \hat{\pi}_1) * (\pi^{(2)} \oplus \hat{\pi}_2) = \pi \oplus \hat{\pi}$. We have $\mu(\sigma_i) = [k_i sr_i(1), \dots, k_i sr_i(l_i)]$. Let $\tilde{\sigma}_i$ denote the restriction of σ_i to $C(A_i)$. From Lemma 5.6 we have

$$\mu(\tilde{\sigma}_i) = [k_i sr_i(1)n_i(1), \dots, k_i sr_i(l_i)n_i(l_i)] = [k_i sn_i, \dots, k_i sn_i].$$

□

The purpose of the next definition and lemma is to emphasize an important property about $*$ -representations satisfying the RCP.

Definition 5.9. A $*$ -representation $\pi : A_1 * A_2 \rightarrow \mathbb{B}(H)$ is said to be *densely perturbable to an irreducible $*$ -representation*, abbreviated *DPI*, if the set

$$\Delta(\pi) := \{u \in \mathbb{U}(H) : \pi^{(1)}(A_1)' \cap (u\pi^{(2)}(A_2)'u^*) = \mathbb{C}\}$$

is norm dense in $\mathbb{U}(H)$. Here the commutants are taken with respect to $\mathbb{B}(H)$.

The next lemma shows that any $*$ -representation satisfying the RCP is DPI.

Lemma 5.10. *Assume A_1 and A_2 are finite dimensional C^* -algebras and $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$. If $\rho : A_1 * A_2 \rightarrow \mathbb{B}(H)$, with H finite dimensional, satisfies the Rank of Central Projections condition, then ρ is DPI.*

Proof. Since $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$, and after interchanging A_1 and A_2 , if necessary, one of the following must hold:

- (1) A_1 and A_2 are simple,
- (2) $\dim C(A_1) \geq 2$ and A_2 is simple,
- (3) for $i = 1, 2$, $A_i = M_{n_i(1)} \oplus M_{n_i(2)}$, with $n_2(2) \geq 2$,
- (4) $\dim C(A_1) \geq 2$, $\dim C(A_2) \geq 3$.

In case (1), take $B_i = \rho^{(i)}(A_i)'$, $i = 1, 2$.

In case (2), let $B_1 = \rho^{(1)}(C(A_1))'$ and $B_2 = \rho^{(2)}(A_2)'$. Notice that $\dim C(B_2) = 1$, $\dim C(B_1) = \dim C(A_1) \geq 2$ and, by the RCP assumption, B_1 is $*$ -isomorphic to $M_{\dim H / \dim C(B_1)} \oplus \cdots \oplus M_{\dim H / \dim C(B_1)}$.

In case (3), let $B_1 = \rho^{(1)}(C(A_1))'$ and $B_2 = \rho^{(2)}(\mathbb{C} \oplus M_{n_2(2)})'$. By the RCP assumption, B_1 is $*$ -isomorphic to

$$M_{\dim H/2} \oplus M_{\dim H/2}$$

and B_2 is $*$ -isomorphic to

$$M_{\dim H/2} \oplus M_{\dim H/(2n_2(2))}.$$

In case (4), let $B_i = \rho^{(i)}(C(A_i))'$ for $i = 1, 2$. Then $\dim C(B_1) = \dim C(A_1) \geq 2$, $\dim C(B_2) = \dim C(A_2) \geq 3$ and, for $i = 1, 2$, RCP implies B_i is $*$ -isomorphic to

$$M_{\dim H / \dim C(B_i)} \oplus \cdots \oplus M_{\dim H / \dim C(B_i)}.$$

Now define

$$\Delta(B_1, B_2) := \{u \in \mathbb{U}(H) : B_1 \cap \text{Ad } u(B_2) = \mathbb{C}\}.$$

and notice that in all four cases $\Delta(B_1, B_2) \subseteq \Delta(\rho)$. By Theorem 4.1, the set $\Delta(B_1, B_2)$ is dense in all the four cases. \square

A downside of the DPI property is that it is not stable under direct sums. However, it is stable under perturbations.

Remark 5.11. If $\pi : A_1 * A_2 \rightarrow \mathbb{B}(H)$ is DPI, then for any u in $\mathbb{U}(H)$, $\pi^{(1)} * (\text{Ad } u \circ \pi^{(2)})$ is also DPI. Indeed, this follows from the identity

$$\Delta(\pi^{(1)} * (\text{Ad } u \circ \pi^{(2)})) = \Delta(\pi)u^*.$$

From Lemma 5.8 we obtain the following.

Lemma 5.12. *For any unital finite dimensional $*$ -representation $\pi : A_1 * A_2 \rightarrow \mathbb{B}(H)$, there is a unital finite dimensional $*$ -representation $\hat{\pi} : A_1 * A_2 \rightarrow \mathbb{B}(\hat{H})$ such that $\pi \oplus \hat{\pi}$ is DPI.*

Proof. The assumption $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$ implies there is a unital finite dimensional $*$ -representation $\vartheta : A_1 * A_2 \rightarrow \mathbb{B}(H_0)$, such that $(\dim(\vartheta^{(1)}(A_1)) - 1)(\dim(\vartheta^{(2)}(A_2)) - 1) \geq 2$. Consider the unital C^* -subalgebras of $\mathbb{B}(H \oplus H_0)$, $D_i = (\pi \oplus \vartheta)^{(i)}(A_i)$, $i = 1, 2$, and notice that $(\dim(D_1) - 1)(\dim(D_2) - 1) \geq 2$. Let $\theta : D_1 * D_2 \rightarrow \mathbb{B}(H \oplus H_0)$ be the unital $*$ -representation induced by the universal property of $D_1 * D_2$ via the unital inclusions $D_i \subseteq \mathbb{B}(H \oplus H_0)$. Lemma 5.8 implies there is a unital finite dimensional $*$ -representation $\rho : D_1 * D_2 \rightarrow \mathbb{B}(K)$ such that $\theta \oplus \rho$ satisfies the RCP condition, so by Lemma 5.10 is DPI.

Let $j_i : D_i \rightarrow D_1 * D_2$, $i = 1, 2$, be the inclusion $*$ -homomorphism from the definition of unital full free product. Now consider the unital $*$ -homomorphism $\sigma = (j_1 \circ (\pi \oplus \vartheta)^{(1)}) * (j_2 \circ (\pi \oplus \vartheta)^{(2)}) : A_1 * A_2 \rightarrow D_1 * D_2$. Now just take $\hat{H} = H_0 \oplus K$ and $\hat{\pi} = \vartheta \oplus (\rho \circ \sigma)$. In order to show $\pi \oplus \hat{\pi}$ is DPI we just need to show that, for $i = 1, 2$, $(\pi \oplus \hat{\pi})^{(i)}(A_i) = (\theta \oplus \rho)^{(i)}(D_i)$, but this is a direct computation. \square

The proof of next lemma is a standard approximation argument and we omit it.

Proposition 5.13. *Let A_1 and A_2 be two unital C^* -algebras. Given a non zero element x in $A_1 * A_2$ and a positive number ε , there is a positive number $\delta = \delta(x, \varepsilon)$ such that for any u and v in $\mathbb{U}(H)$ satisfying $\|u - v\| < \delta$ and any unital $*$ -representations $\pi : A_1 * A_2 \rightarrow \mathbb{B}(H)$, we have*

$$\|(\pi^{(1)} * (\text{Ad } v \circ \pi^{(2)}))(x) - (\pi^{(1)} * (\text{Ad } u \circ \pi^{(2)}))(x)\| < \varepsilon.$$

Here is our main theorem.

Theorem 5.14. *Assume A_1 and A_2 are unital, separable, residually finite dimensional C^* -algebras with $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$. Then $A_1 * A_2$ is primitive.*

Proof. By the result of Exel and Loring in [7], there is a separating sequence $(\pi_j : A_1 * A_2 \rightarrow \mathbb{B}(H_j))_{j \geq 1}$, of finite dimensional unital $*$ -representations. For later use in constructing an essential representation of $A_1 * A_2$, i.e., a $*$ -representation with the property that zero is the only compact operator in its image, we modify $(\pi_j)_{j \geq 1}$, if necessary, so that that each $*$ -representation is repeated infinitely many times.

By recursion and using Lemma 5.12, we define a sequence

$$\hat{\pi}_j : A_1 * A_2 \rightarrow \mathbb{B}(\hat{H}_j), \quad (j \geq 1)$$

of finite dimensional unital $*$ -representations such that, for all $k \geq 1$, $\oplus_{j=1}^k (\pi_j \oplus \hat{\pi}_j)$ is DPI. Let $\pi := \oplus_{j \geq 1} \pi_j \oplus \hat{\pi}_j$ and $H := \oplus_{j \geq 1} H_j \oplus \hat{H}_j$. To ease notation, for $k \geq 1$, let $\pi_{[k]} = \oplus_{j=1}^k \pi \oplus \hat{\pi}$. Note that we have $\pi(A_1 * A_2) \cap \mathbb{K}(H) = \{0\}$. Indeed, if $\pi(x)$ is compact then $\lim_j \|(\pi_j \oplus \hat{\pi}_j)(x)\| = 0$, since each representation is repeated infinitely many times and we are considering a separating family we get $x = 0$.

We will show that given any positive number ε , there is a unitary u on $\mathbb{U}(H)$ such that $\|u - \text{id}_H\| < \varepsilon$ and $\pi^{(1)} * (\text{Ad } u \circ \pi^{(2)})$ is both irreducible and faithful. To do this, we will to construct a sequence $(u_k, \theta_k, F_k)_{k \geq 1}$ where:

- (a) For all k , u_k is a unitary in $\mathbb{U}(\oplus_{j=1}^k H_j \oplus \hat{H}_j)$ satisfying

$$\|u_k - \text{id}_{\oplus_{j=1}^k H_j \oplus \hat{H}_j}\| < \frac{\varepsilon}{2^{k+1}}. \quad (18)$$

- (b) Letting

$$u_{(j,k)} = u_j \oplus \text{id}_{H_{j+1} \oplus \hat{H}_{j+1}} \oplus \cdots \oplus \text{id}_{H_k \oplus \hat{H}_k}$$

and

$$U_k = u_k u_{(k-1,k)} u_{(k-2,k)} \cdots u_{(1,k)}, \quad (19)$$

the unital $*$ -representation of $A_1 * A_2$ onto $\mathbb{B}(\oplus_{j=1}^k H_j \oplus \hat{H}_j)$, given by

$$\theta_k = \pi_{[k]}^{(1)} * (\text{Ad } U_k \circ \pi_{[k]}^{(2)}), \quad (20)$$

is irreducible.

- (c) F_k is a finite subset of the closed unit ball of $A_1 * A_2$ and for all y in the closed unit ball of $A_1 * A_2$ there is an element x in F_k such that

$$\|\theta_k(x) - \theta_k(y)\| < \frac{1}{2^{k+1}}. \quad (21)$$

- (d) If $k \geq 2$, then for any element x in the union $\cup_{j=1}^{k-1} F_j$, we have

$$\|\theta_k(x) - (\theta_{k-1} \oplus \pi_k \oplus \hat{\pi}_k)(x)\| < \frac{1}{2^{k+1}}. \quad (22)$$

We construct such a sequence by recursion.

Step 1: Construction of (u_1, θ_1, F_1) . Since $\pi \oplus \hat{\pi}$ is DPI, there is a unitary u_1 in $H_1 \oplus \hat{H}_1$ such that $\|u_1 - \text{id}_{H \oplus \hat{H}}\| < \frac{\varepsilon}{2^2}$ and $\pi_{[1]}^{(1)} * \text{Ad } u_1 \circ \pi_{[1]}^{(2)}$ is irreducible. Hence condition (18) and (20) trivially hold. Since $H_1 \oplus \hat{H}_1$ is finite dimensional, there is a finite set F_1 contained in the closed unit ball of $A_1 * A_2$ satisfying condition (21). At this stage there is no condition (22).

Step 2: Construction of $(u_{k+1}, \theta_{k+1}, F_{k+1})$ from (u_j, θ_j, F_j) , $1 \leq j \leq k$. First, we prove there exists a unitary u_{k+1} in $\mathbb{U}(\oplus_{j=1}^{k+1} H_j \oplus \hat{H}_j)$ such that $\|u_{k+1} - \text{id}_{\oplus_{j=1}^{k+1} H_j \oplus \hat{H}_j}\| < \frac{\varepsilon}{2^{k+2}}$, the unital $*$ -representation of $A_1 * A_2$ into $\mathbb{B}(\oplus_{j=1}^{k+1} H_j \oplus \hat{H}_j)$ defined by

$$\theta_{k+1} := (\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1})^{(1)} * (\text{Ad } u_{k+1} \circ (\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1})^{(2)}) \quad (23)$$

is irreducible and for any element x in the union $\cup_{j=1}^k F_j$, the inequality $\|\theta_{k+1}(x) - (\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1})(x)\| < \frac{1}{2^{k+1}}$, holds. By Remark 5.11, $\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1}$ is DPI so Proposition 5.13 assures the existence of such unitary u_{k+1} . Notice that, from construction, conditions (18) and (22) are satisfied. A consequence of (20) and (19) is

$$\theta_{k+1} = \pi_{[k+1]}^{(1)} * (\text{Ad } U_{k+1} \circ \pi_{[k+1]}^{(2)}).$$

Finite dimensionality of $\oplus_{j=1}^{k+1} H_j \oplus \hat{H}_j$ guarantees the existence of a finite set F_{k+1} contained in the closed unit ball of $A_1 * A_2$ satisfying condition (21). This completes Step 2.

Now consider the $*$ -representations

$$\sigma_k = \theta_k \oplus \bigoplus_{j \geq k+1} \pi_j \oplus \hat{\pi}_j. \quad (24)$$

We now show there is a unital $*$ -representation of $\sigma : A_1 * A_2 \rightarrow \mathbb{B}(H)$, such that for all x in $A_1 * A_2$, $\lim_k \|\sigma_k(x) - \sigma(x)\| = 0$. If we extend the unitaries u_k to all of H via $\tilde{u}_k = u_k \oplus_{j \geq k+1} \text{id}_{H_j \oplus \hat{H}_j}$, then we obtain

$$\sigma_k = \pi^{(1)} * (\text{Ad } \tilde{U}_k \circ \pi^{(2)}), \quad (25)$$

where $\tilde{U}_k = \tilde{u}_k \cdots \tilde{u}_1$. Thanks to condition (18), we have

$$\|\tilde{U}_k - \text{id}_H\| \leq \sum_{j=1}^k \|\tilde{u}_k - \text{id}_H\| < \sum_{j=1}^k \frac{\varepsilon}{2^{k+1}},$$

and for $l \geq 1$

$$\|\tilde{U}_{k+l} - \tilde{U}_k\| = \|\tilde{u}_{k+l} \cdots \tilde{u}_{k+1} - \text{id}_H\| \leq \sum_{j=k+1}^{k+l} \frac{\varepsilon}{2^{j+1}}.$$

Hence, Cauchy's criterion implies there is a unitary u in $\mathbb{U}(H)$ such that the sequence $(\tilde{U}_k)_{k \geq 1}$ converges in norm to u and $\|u - \text{id}_H\| < \frac{\varepsilon}{2}$. Define

$$\sigma = \pi^{(1)} * (\text{Ad } u \circ \pi^{(2)}). \quad (26)$$

From Proposition 5.13 we have that for all x in $A_1 * A_2$,

$$\lim_k \|\sigma_k(x) - \sigma(x)\| = 0. \quad (27)$$

Our next goal is to show σ is irreducible. To ease notation let $A = A_1 * A_2$. We will show $\overline{\sigma(A)}^{SOT} = \mathbb{B}(H)$. Take T in $\mathbb{B}(H)$. With no loss of generality we may assume $\|T\| \leq \frac{1}{2}$. Recall that a neighborhood basis for the SOT topology around T is given by the sets

$$\mathcal{N}_T(\xi_1, \dots, \xi_n; \varepsilon) = \{S \in \mathbb{B}(H) : \|S\xi_i - T\xi_i\| < \varepsilon, i = 1, \dots, n\}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$, and $\xi_1, \dots, \xi_n \in H$ are unit vectors. We show that for any $\varepsilon > 0$ and any unit vectors ξ_1, \dots, ξ_n , $\mathcal{N}_T(\xi_1, \dots, \xi_n; \varepsilon) \cap \sigma(A)$ is nonempty. Let P_k denote the orthogonal projection from H onto $\oplus_{j=1}^k H_j \oplus \hat{H}_j$. Take $k_1 \geq 1$ such

$$\sum_{k \geq k_1} \frac{1}{2^k} < \frac{\varepsilon}{2^3}$$

and for $k \geq k_1$, $1 \leq i \leq n$,

$$\|(\text{id}_H - P_k)(\xi_i)\| < \frac{\varepsilon}{2^3}, \quad (28)$$

$$\|(\text{id}_H - P_k)(T\xi_i)\| < \frac{\varepsilon}{2^3}. \quad (29)$$

Since P_k has finite rank and θ_k is irreducible, there is a in A , with $\|a\| \leq 1$ such that

$$P_{k_1} T P_{k_1}(\xi_i) = \theta_{k_1}(a)(P_{k_1}(\xi_i)) \quad (30)$$

for $i = 1, \dots, n$. We have

$$\theta_{k_1}(a)(P_{k_1}(\xi_i)) = \sigma_{k_1}(a)(P_{k_1}(\xi_i)). \quad (31)$$

Take x in F_{k_1} such that

$$\|\theta_{k_1}(a) - \theta_{k_1}(x)\| < \frac{1}{2^{k_1+1}}. \quad (32)$$

We will show $\sigma(x) \in \mathcal{N}_T(\xi_1, \dots, \xi_n; \varepsilon)$. To ease notation let $\xi_i = \xi$. From (28), (29), (30) and (31), we deduce

$$\begin{aligned} \|T\xi - \sigma(x)\xi\| &\leq \|T\xi - P_{k_1}TP_{k_1}\xi\| \\ &\quad + \|P_{k_1}TP_{k_1}\xi - \sigma_{k_1}(a)\xi\| \\ &\quad + \|\sigma_{k_1}(a)\xi - \sigma(x)\xi\| \\ &< \frac{3\varepsilon}{2^3} + \|\sigma_{k_1}(a)\xi - \sigma(x)\xi\|. \end{aligned}$$

For any $p \geq 1$ we have

$$\begin{aligned} \sigma_{k_1}(a)\xi - \sigma(x)\xi &= \sigma_{k_1}(a)\xi - \sigma_{k_1}(x)\xi \\ &\quad + \sum_{j=k_1}^{k_1+p} (\sigma_j(x)\xi - \sigma_{j+1}(x)\xi) \\ &\quad + \sigma_{k_1+p+1}(x)\xi - \sigma(x)\xi. \end{aligned}$$

Thus, from (28), (31), (32), (24) and (22) we deduce

$$\|\sigma_{k_1}(a)\xi - \sigma(x)\xi\| < \frac{\varepsilon}{2} + \|\sigma_{k_1+p+1}(x)\xi - \sigma(x)\xi\|$$

hence

$$\|\sigma_{k_1}(a)\xi - \sigma(x)\xi\| \leq \frac{\varepsilon}{2}.$$

We conclude $\sigma(x)$ lies in $\mathcal{N}_T(\xi_1, \dots, \xi_n; \varepsilon)$.

An application of Choi's technique (see Theorem 6 in [4]) will give us faithfulness of σ . Indeed, from construction, for all x in A , $\sigma(x) = \lim_k \sigma_k(x)$. Thus if each σ_k is faithful then so is σ . But faithfulness of σ_k follows from the commutativity of the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \mathbb{B}(H) \\ \sigma_k \downarrow & & \downarrow \pi_C \\ \mathbb{B}(H) & \xrightarrow{\pi_C} & \mathbb{B}(H)/\mathbb{K}(H) \end{array}$$

(where π_C denotes the quotient map onto the Calkin algebra), which in turn is implied by (24). \square

To obtain the following corollary, see Lemma 3.2 of [1].

Corollary 5.15. *Assume A_1 and A_2 are nontrivial residually finite dimensional C^* -algebras with $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$. Then $A_1 * A_2$ is antiliminal and has an uncountable family of pairwise inequivalent irreducible faithful $*$ -representations.*

We finish with a corollary derived from Lemma 11.2.4 in [6].

Corollary 5.16. *Assume A_1 and A_2 are nontrivial residually finite dimensional C^* -algebras with $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$. Then pure states of $A_1 * A_2$ are w^* -dense in the state space.*

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